DIE BÖSE FARBE

A. BAUDISCH, M. HILS, A. MARTIN-PIZARRO AND F. O. WAGNER

ABSTRACT. We construct a bad field in characteristic zero. That is, we construct an algebraically closed field which carries a notion of dimension analogous to Zariski-dimension, with an infinite proper multiplicative subgroup of dimension one, and such that the field itself has dimension two. This answers a longstanding open question by Zilber.

1. Introduction

Morley rank is a model-theoretical generalization of *Zariski dimension* which can be extended to definable sets in any mathematical structure. A structure is called ω-stable if Morley rank is ordinal-valued. Morley degree is the analogous to the number of irreducible components of maximal Zariski dimension. An early conjecture due to B. Zilber stated that a structure of both Morley rank and degree 1 (called a *strongly minimal set*) arose from a classical geometry: the trivial (or degenerated) one, a vector space geometry or Zariski geometry over an algebraically closed field. This conjecture was refuted in 1988 by E. Hrushovski [11], who modified in a clever way Fraïssé's original construction of the universal homogeneous model of a hereditary class of finite relational structures with the amalgamation property. This method was later divided by B. Poizat and J. B. Goode [8] into two steps: first, the construction of a generic structure of rank ω and secondly, the *collapse* to a strongly minimal set. This procedure has been used in several applications: for example, E. Hrushovski *fused* two strongly minimal sets with definable Morley degree in disjoint languages into a strongly minimal set [10] (*cf.* also [3]); it follows in particular that there exist a strongly minimal set which supports two field structures (even in different characteristics) with as little interaction as possible. In the aforementioned article, he also mentioned that this procedure could also be generalized to the case where both strongly minimal sets were expansions of a common vector space structure over a finite field. The fusion over a vector space was first proved by the second author and A. Hasson [9] in the 1-based case (moreover, they also studied the non-collapsed fusion of rank ω). The collapse to a strongly minimal fusion was attained finally by the first and third authors together with M. Ziegler [4]. Using similar arguments they also proved [5] the existence of a field of arbitrary prime characteristic of Morley rank 2 equipped with a definable additive subgroup of rank 1 after collapsing Poizat's *red fields* [16] of rank $\omega \cdot 2$. In

Date: 6. November 2007.

1991 *Mathematics Subject Classification.* Primary: 03C65; Secondary: 03C50.

Key words and phrases. Model Theory, Fields of finite Morley Rank.

The first and the fourth author were supported by the European Research Training Network FP6-MRTN-CT-2004-512234 Modnet, and the third author was supported by the Marie Curie postdoctoral fellowship FP6-MOB-5-009541.

The fourth author is a junior member of the Institut Universitaire de France.

particular, there is a strongly minimal set with one addition and two independent multiplications.

Bad fields are fields of finite Morley rank equipped with a predicate for a non-trivial proper divisible multiplicative subgroup. They appeared first in the study of simple groups of finite Morley rank, whose Borel subgroups (i.e. maximal solvable subgroups) are of the form $K^+ \rtimes T$ with $1 < T < K^\times$. According to the *algebraicity conjecture* due to Cherlin-Zil'ber (an algebraic variation of Zilber's above conjecture) a simple infinite group of finite Morley rank is algebraic. The non-existence of bad fields would simplify the study of Borel subgroups. Due to a result by the fourth author [17, 18] bad fields are unlikely to exist in positive characteristic. After applying deep results due to J. Ax [1], B. Poizat [16] found a candidate of rank $\omega \cdot 2$ with a multiplicative subgroup of rank ω based on his construction [15] of a field of rank $\omega \cdot 2$ equipped with a definable set of rank ω; the latter is a counterexample to a conjecture due to C. Berline and D. Lascar, which asserts that the rank of a field (if ordinal) should be a monomial of the form ω^α. Using amalgamation, Poizat [15] also obtained collapsed structures. J. Baldwin and K. Holland showed in [2] that these structures are ω-saturated under certain assumptions and thus ω-stable of Morley rank 2. Together with M. Ziegler, the first and third author summarized and completed [6] the above results and exhibited a simpler axiomatization. Let us remark that a field of ordinal Morley rank is algebraically closed [13].

Following Poizat's notation we denote the predicate for the multiplicative subgroup by *green* (an additive subgroup is *red*, and a subset *black*). In this article we collapse the green fields and therefore obtain a bad field in characteristic 0. This construction is extremely related to the red collapse [5]; however we will use results due to Ax-Poizat instead of locally finiteness of $\mathbb{F}_p$-vector spaces.

We thank the referee for his detailed comments and for his questions, too. Unfortunately, we were not able to reply to all of them. This is ongoing work.

2. Algebraic Lemmata

This section summarizes results coming from algebraic geometry which will be needed for our purposes. Let us fix some notation: $\mathbb{C}$ denotes an algebraically closed field of characteristic 0. A variety V will always be a subvariety of some cartensian product $(\mathbb{C}^*)^n$. A *torus* is a connected algebraic subgroup of $(\mathbb{C}^*)^n$. It is described by finitely many equations of the form: $x_1^{r_1} \cdot \ldots \cdot x_n^{r_n} = 1$. Linear dimension (as $\mathbb{Q}$-vector spaces modulo torsion) equals algebraic dimension (as varieties) for tori which we will denote as $\mathrm{l.dim}_{\mathbb{Q}}(T)$ or $\dim(T)$. Given a closed irreducible subvariety V in $(\mathbb{C}^*)^n$, its *minimal torus* is the smallest torus T such that V lies in some coset $\bar{a} \cdot T$ (with $\bar{a} \in V$). In this case, we define the *codimension* of an irreducible variety V as $\mathrm{cd}(V) := \dim(T) - \dim(V) = \mathrm{l.dim}_{\mathbb{Q}}(V) - \dim(V)$, where $\mathrm{l.dim}_{\mathbb{Q}}(V) := \dim(T)$. A subvariety $W \subseteq V$ is cd-*maximal* if $\mathrm{cd}(W') > \mathrm{cd}(W)$ for every subvariety $W \subsetneq W' \subseteq V$. Clearly, irreducible components of V and tori cosets maximally contained in V are examples of cd-maximal subvarieties.

Fact 2.1. A connected algebraic subgroup of a torus is again a torus.

We now state a result proved by B. Poizat [16, Corollaires 3.6 und 3.7]:

Theorem 2.2. *Let $V(\bar{x}, \bar{z})$ be a uniformly definable family of varieties in $(\mathbb{C}^*)^n$. There exists a finite collection of tori $\{T_0, \ldots, T_r\}$, such that for any torus $T \subseteq (\mathbb{C}^*)^n$, any member $V_{\bar{b}} = V(\bar{x}, \bar{b})$ of the family and any irreducible component W*

of $V_{\bar{b}} \cap \bar{a} \cdot T$ *(with* $\bar{a}$ *in* W*) there is some* i *in* $\{0, \ldots, r\}$ *with* $W \subseteq \bar{a} \cdot T_i$ *and* $\dim(T_i) - \dim(V \cap \bar{a} \cdot T_i) \dim T - \dim W$.

Moreover, the minimal torus of every cd*-maximal subvariety of* $V_{\bar{b}}$ *belongs to the collection* $\{T_0, \ldots, T_r\}$.

We will assume throughout this article that the above tori are all distinct, and $T_0 = (\mathbb{C}^*)^n$ and $T_1 = \{1\}^n$.

Note 2.3. Only the second claim of theorem 2.2 is needed for our purposes.

Note 2.4. The content of 2.2 is false in positive characteristic. We thank M. Bays and B. Zilber for the following example. Let K be an algebraically closed field of characteristic $p > 0$ and let $V \subseteq (K^*)^4$ be defined by the equations $x + y = 1$ and $z + u = 1$. For $n \in \mathbb{N}$ let $T_n \subseteq (K^*)^4$ be the torus defined by the equations $z^{p^n} = x$ and $u^{p^n} = y$. It is easy to show:

- V is irreducible, with $\dim(V) = 2$ and $\mathrm{cd}(V) = 2$;
- $X_n := V \cap T_n$ is an irreducible curve with minimal torus T_n, and $\mathrm{cd}(X_n) = 1$ holds;
- X_n is a cd-maximal subvariety of V.

A family of tori with the properties as in in 2.2 would necessarily contain all tori T_n, which are all distinct.

Corollary 2.5. *Let* $V(\bar{x}, \bar{z})$ *be a uniformly definable family of varities. Then:*

1. *If* T *is the minimal torus of* $V_{\bar{b}}$*, there exists some* $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$*, such that* T *is the minimal torus of every* $V_{\bar{b}'}$ *with* $\models \theta(\bar{b}')$*. In particular, there is a definable neighbourhood of* $\bar{b}$ *where* $\mathrm{l.dim}_{\mathbb{Q}}$ *and* cd *remain constant.*
2. *Suppose* $V(\bar{x}, \bar{b})$ *decomposes into* m *irreducible components* W_k *with* $d_k := \dim(W_k)$*,* $l_k := \mathrm{l.dim}_{\mathbb{Q}}(W_k)$ *and* $c_k := \mathrm{cd}(W_k)$*. Then, there is some* $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$ *such that for all* $\bar{b}' \models \theta$ *we have that* $V_{\bar{b}'}$ *decomposes into exactly* m *irreducible components* $(W'_k : 1 \leq k \leq m)$ *and (possibly after permutation)* $\dim(W'_k) = d_k$*,* $\mathrm{l.dim}_{\mathbb{Q}}(W'_k) = l_k$ *and* $\mathrm{cd}(W'_k) = c_k$*.*

Proof. The first claim follows from 2.2, since irreducible components are cd-maximal. So their minimal tori lie in $\{T_0, \ldots, T_r\}$.

It is a classical result (see e.g. [7]) that the decomposition of a variety in its irreducible components is definable. Part (2) thus follows from part (1). □

3. δ-Arithmetic

Let now $\mathbb{C}$ denote a large algebraically closed field of characteristic 0, i.e. a universal model of ACF_0. We consider the reduct $\mathcal{L}_{mult} := \{\cdot, 1, 0, =\} \subseteq \mathcal{L}_{Ring} = \{+, \cdot, 1, 0, =\}$ and $T_{mult} := Th_{\mathcal{L}_{mult}}(\mathbb{C})$. The prime model of T_{mult} is obtained by adjoining 0 to the set of roots of unity $\mu(\mathbb{C}) \subseteq \mathbb{C}$. The structure so obtained is $\mathcal{L}_{mult}$-isomorphic to $\mathbb{Q}/\mathbb{Z}$ (multiplicatively), after adding a new element. In particular, T_{mult} is modular non-trivial, and its geometry is projective over $\mathbb{Q}$.

Given $A \subseteq \mathbb{C}$, denote by $\langle A \rangle$ the divisible hull of $A^* := A \setminus \{0\}$ in $\mathbb{C}^*$ together with 0, equivalently, the algebraic closure of A in $\mathbb{C}$ with respect to $\mathcal{L}_{mult}$. This yields the prime model of T_{mult} over A. For a tuple $\bar{a} \in \mathbb{C}^*$ we have that $MR\ (\bar{a})$ in T_{mult} agrees with its linear dimension over $\mathbb{Q}$ (modulo torsion). We will write $\mathrm{l.dim}_{\mathbb{Q}}(\bar{a})$, and let $\dim(\bar{a}/B)$ denote Morley rank in ACF_0, equivalently, the dimension of its locus over B. The latter agrees with the transcendence degree of $\bar{a}$ over B.

Consider now $\langle\cdot\rangle$-closed subsets B of $\mathbb{C}$ as an $\mathcal{L}_{Ring}$-structure. Generally these structures need not be subfields of $\mathbb{C}$, however we will (after a possible Morleyzation $\mathcal{L}_{Morley}$ of ACF_0 consisting of $\mathcal{L}_{mult}$ and relational symbols) formally work with these structures. Therefore, we will consider addition on $\langle\cdot\rangle$-closed sets and refer to their fraction fields, their algebraic closure, etc. However this shall not confuse the unfortunate reader. Let $\mathcal{D}$ be the class of such structures as above described. Embeddings in $\mathcal{D}$ are elementary embeddings as $\mathcal{L}_{Ring}$-structures, i.e. $\mathcal{L}_{Morley}$-embeddings. A structure $A \in \mathcal{D}$ is *finitely generated* over $B \subseteq A$ if $A = \langle \bar{a}B\rangle$ for some finite tuple $\bar{a} \in A$. Finitely generated elements of $\mathcal{D}$ over B correspond to divisible hulls of subgroups of $\mathbb{C}^*$ of finite rank over $\langle B\rangle$ (adjoining 0). In case that $\langle AB\rangle$ is finitely generated over $\langle B\rangle$, we define

$$\delta(A/B) := 2\dim(A/B) - \mathrm{l.\,dim}_{\mathbb{Q}}(A/B).$$

Clearly, $\delta(A/B)$ equals $\delta(\langle AB\rangle/\langle B\rangle)$. Since T_{mult} is modular, the following holds:

Lemma 3.1.

(1) $\delta(\bar{a}\bar{b}/C) = \delta(\bar{b}/C) + \delta(\bar{a}/\bar{b}C)$.
(2) Given $C \subseteq B \in \mathcal{D}$, we have that $\delta(\bar{a}/B) \leq \delta(\bar{a}/B \cap \langle C\bar{a}\rangle)$. (Submodularity)

Lemma 3.2. *Given $\bar{a}$ and B, let W be the locus of $\bar{a}$ over* $\mathrm{acl}(B)$. *Then:*

$$\delta(\bar{a}/\,\mathrm{acl}(B)) = \dim(W) - \mathrm{cd}(W)$$

Moreover,

$$\delta(\bar{a}/B) = \dim(W) - \mathrm{cd}(W) - \mathrm{l.\,dim}_{\mathbb{Q}}(\langle \bar{a}B\rangle \cap \mathrm{acl}(B)/B).$$

Proof. For the first claim it suffices to observe that the smallest torus coset (over an algebraically closed set B) containing $\bar{a}$ is B-definable in $\mathcal{L}_{mult}$. Hence, its dimension equals $\mathrm{l.\,dim}_{\mathbb{Q}}(\bar{a}/B)$. Modularity of T_{mult} gives the second statement. □

Definition 3.3.

- Let $M \subseteq N \in \mathcal{D}$ with $\mathrm{l.\,dim}_{\mathbb{Q}}(N/M) = n \geq 2$. The extension N/M is *minimal prealgebraic* (of *length* n) if $\delta(N/M) = 0$ and $\delta(N'/M) > 0$ for every $N' = \langle N'\rangle \in \mathcal{D}$ with $M \subsetneq N' \subsetneq N$.
- Let $B \subseteq \mathbb{C}$. A strong type $p(\bar{x}) \in S^n(B)$ (in ACF_0) is *minimal prealgebraic*, if the extension $\langle B\bar{a}\rangle/\langle B\rangle$ is minimal prealgebraic of length n for some $\bar{a} \models p$ (in particular, $\bar{a}$ is multiplicatively independent over B).
- A formula $\varphi(\bar{x})$ of Morley degree 1 is *minimal prealgebraic* if its generic type is minimal prealgebraic.

Note 3.4.

- The condition $\delta(N'/M) > 0$ is equivalent to $\delta(N/N') < 0$ since $0 = \delta(N/M) = \delta(N/N') + \delta(N'/M)$.
- If N/M is minimal prealgebraic and $\bar{n}$ is a multiplicative basis of N over M, then $\mathrm{stp}(\bar{n}/M)$ is minimal prealgebraic.
- Minimal prealgebraicity is invariant under parallelism class and multiplicative translation for strong types. In particular, the notion of minimal prealgebraic for stationary formulae in Definition 3.3 is well-defined.

4. Codes

We first aim to encode minimal prealgebraic extensions. Note that every strong type is the generic type of some variety. Hence, we can define the following:

Definition 4.1. A variety $V = V(\bar{x}, \bar{b})$ is a *code variety* if it is minimal prealgebraic. Equivalently, if for all $B = \langle B \rangle \ni \bar{b}$ the extension $B \subseteq \langle B\bar{a} \rangle$ is minimal prealgebraic for some B-generic $\bar{a} \in V$.

Note that a multiplicative translation of a code variety is again a code variety.

Let $M \in \mathcal{D}$ and $N = \langle M\bar{a} \rangle$ where $\mathrm{tp}(\bar{a}/M)$ is minimal prealgebraic of length n. Consider $N' = \langle N' \rangle$ with $M \subsetneq N' \subsetneq N$. Modularity of T_{mult} yields some $\mathcal{L}_{mult}$-basis $\bar{a}' \in \langle \bar{a} \rangle$ for N' over M of length m. Modulo torsion we have that $a'_j = \prod a_i^{\lambda_{ij}}$ for some $\lambda_{ij} \in \mathbb{Q}$. After substitution with suitable powers or roots, we may assume that $\lambda_{ij} \in \mathbb{Z}$ and $(\lambda_{ij})_{i<n}$ are coprime for $j < m$. Then, the equations $(\prod_{i<n} x_i^{\lambda_{ij}} = 1 : j < m)$ determine a torus T of dimension $d = n - m$. Since T is $\emptyset$-definable, it follows that $\bar{a}'$ is the canonical basis $[\bar{a}T]$ of the coset $\bar{a}T$.

Similarly, given a torus T in $(\mathbb{C}^*)^n$ of $\dim(T) = d$, the element $\bar{a}' := [\bar{a}T]$ generates a substructure $N' := \langle M\bar{a}' \rangle \subseteq N$ with $\mathrm{l.dim}_{\mathbb{Q}}(N'/M) = m = n - d$.

Lemma 4.2. *Let $V(\bar{x}, \bar{b}) \subseteq \mathbb{C}^n$ be an irreducible variety, $T \subseteq \mathbb{C}^n$ some torus and $\bar{a}$ in V generic over $B \ni \bar{b}$. Then, an element $\bar{a}_1 \in W := V \cap \bar{a}T$ is generic in W over $B[\bar{a}T]$ if and only if it is generic in $V_{\bar{b}}$.*

Proof. Note that $\bar{a}' := [\bar{a}T]$ is definable multiplicatively over any $\bar{a}_2 \in \bar{a}T$. In particular, it is definable over $\bar{a}_1 \in W$. Hence:

$$\begin{aligned}\dim(\bar{a}_1/B\bar{a}') &= \dim(\bar{a}_1\bar{a}'/B) - \dim(\bar{a}'/B)\dim(\bar{a}_1/B) - \dim(\bar{a}'/B)\\ &\leq \dim(\bar{a}/B) - \dim(\bar{a}'/B) = \dim(\bar{a}/B\bar{a}'),\end{aligned}$$

which proves the claim. □

The following result was already stated in [16]; however we exhibit a proof for completeness since similar ideas will appear later on.

Lemma 4.3. *Let $V(\bar{x}, \bar{z})$ be a uniformly definable family of varieties. If $V_{\bar{b}} = V(\bar{x}, \bar{b})$ is a code variety then there is a formula $\theta(\bar{z}) \in \mathrm{tp}(\bar{b})$ such that $V_{\bar{b}_1}$ is a code variety for every $\bar{b}_1 \models \theta$.*

Proof. Let $\{T_0, \ldots, T_r\}$ be the collection of tori associated to $V(\bar{x}, \bar{z})$ as in 2.2. Take some B containing $\bar{b}$. Set

$$\begin{aligned} n &= \mathrm{l.dim}_{\mathbb{Q}}(V_{\bar{b}}) = 2k\\ k &= \dim(V_{\bar{b}}).\end{aligned}$$

Clearly $\langle B\bar{g} \rangle \cap \mathrm{acl}(B) = \langle B \rangle$ for B–generic $\bar{g} \in V_{\bar{b}}$.

Let $\theta(\bar{z})$ express:

(1) $\dim(V_{\bar{z}}) = k$ and $\mathrm{l.dim}_{\mathbb{Q}}(V_{\bar{z}}) = n$ (in particular $V_{\bar{z}} \neq \emptyset$),
(2) given generic $\bar{g}$ in $V_{\bar{z}}$, $i = 2, \ldots, r$ and W some irreducible component of $V \cap \bar{g}T_i$ of maximal dimension, then $\mathrm{cd}(W) > \dim(W)$ if $V \cap \bar{g}T_i$ is infinite.

The existence of such a formula θ follows from Corollary 2.5.

Now we show that $\models \theta(\bar{b})$. Let $\bar{g}$ in $V_{\bar{b}}$ be generic and take $T \neq T_0, T_1$ some torus with $V \cap \bar{g}T$ infinite. Choose some irreducible component $W \subseteq V \cap \bar{g}T$ of maximal dimension. By Lemma 4.2 we have that $\bar{g}$ is generic in W over $\mathrm{acl}((\bar{b}, [\bar{g}T])$. Hence,

$\bar{g} \notin \operatorname{acl}(\bar{b}, [\bar{g}T])$ and $[\bar{g}T] \notin \operatorname{acl}(\bar{b})$. By Lemma 3.2 and miminal prealgebraicity of the extension $\langle \bar{g}\bar{b}\rangle / \langle \bar{b}\rangle$

$$\dim(W) - \operatorname{cd}(W) = \delta(\langle \bar{g}\bar{b}\rangle / \operatorname{acl}(\bar{b}, [\bar{g}T])) = \delta(\langle \bar{g}\bar{b}\rangle / \operatorname{acl}(\bar{b}, [\bar{g}T]) \cap \langle \bar{g}\bar{b}\rangle)) < 0.$$

Now let $\bar{b}_1 \models \theta$ and $\bar{g}$ generic in $V_{\bar{b}_1}$ over $B_1 \ni b_1$. Condition (1) in $\theta(\bar{z})$ yields $\delta(\bar{g}/B_1) = 0$. We need only show that $\delta(\bar{g}/[\bar{g}T], B_1) < 0$ for every torus T with $1 \leq d := \dim(T) \leq n-1$. Set $\bar{g}' := [\bar{g}T]$ and let W be the locus of $\bar{g}$ over $\operatorname{acl}(B_1\bar{g}')$. We consider three cases:

Case 1: $\bar{g} \in \operatorname{acl}(B_1\bar{g}')$, i.e. W is a point. Then

$$\begin{aligned}\delta(\bar{g}/B_1\bar{g}') &= -\operatorname{l.dim}_{\mathbb{Q}}(\bar{g}/B_1\bar{g}')\operatorname{l.dim}_{\mathbb{Q}}(\bar{g}'/B_1) - \operatorname{l.dim}_{\mathbb{Q}}(\bar{g}\bar{g}'/B_1)\\ &= \operatorname{l.dim}_{\mathbb{Q}}(\bar{g}'/B_1) - \operatorname{l.dim}_{\mathbb{Q}}(\bar{g}/B_1) = (n-d) - n = -d < 0.\end{aligned}$$

Case 2: $\operatorname{cd}(W) = \operatorname{cd}(V_{\bar{b}_1})$. Since $W \subsetneq V_{\bar{b}_1}$ and by Lemma 3.2

$$\delta(\bar{g}/B_1\bar{g}') \leq \dim(W) - \operatorname{cd}(W) < \dim(V_{\bar{b}_1}) - \operatorname{cd}(V_{\bar{b}_1}) = 0.$$

Case 3: W is infinite and $\operatorname{cd}(W) < \operatorname{cd}(V_{\bar{b}_1})$. Choose some $W \subseteq \tilde{W} \subseteq V_{\bar{b}_1}$ irreducible maximal such that $\operatorname{cd}(\tilde{W}) \leq \operatorname{cd}(W)$. Then, $\tilde{W}$ is cd-maximal with minimal Torus T_i in the above collection. Note that $i \neq 0$ because $\operatorname{cd}(W) < \operatorname{cd}(V_{\bar{b}_1})$, and $i \neq 1$ because W is infinite. So, $\tilde{W} \subseteq V \cap \bar{g}T_i$. Take now some irreducible component $\tilde{W} \subset \tilde{W}'$ of $V \cap \bar{g}T_i$ of maximal dimension. By cd-maximality $\operatorname{cd}(\tilde{W}') > \dim(\tilde{W}')$. So

$$\delta(\bar{g}/B_1\bar{g}') \leq \dim(W) - \operatorname{cd}(W) \leq \dim(\tilde{W}) - \operatorname{cd}(\tilde{W}) \leq \dim(\tilde{W}') - \operatorname{cd}(\tilde{W}') < 0.$$

□

The above proof points out how to find — uniformly — a generic (definable) subset $\varphi_1(\bar{x}, \bar{b})$ of a code variety $V(\bar{x}, \bar{b})$ such that $\varphi_1(\bar{x}, \bar{b})$ is strongly minimal in the theory T_ω of the generic (non-collapsed) green field in case we *color* $\bar{x}$ green. Given a uniformly definable family $V(\bar{x}, \bar{z})$ of code varieties whose associated tori are $\{T_0, \ldots, T_r\}$ as in 2.2, we can define $\varphi_1(\bar{x}, \bar{z}) \subseteq V(\bar{x}, \bar{z})$ as follows:

- $\bar{a} \in V_{\bar{b}}$ realizes $\varphi_1(\bar{a}, \bar{b})$ if and only if for $i = 2, \ldots, r$ the following condition holds: In case $V_{\bar{b}} \cap \bar{a}T_i$ is infinite then $\operatorname{cd}(W) > \dim(W)$ for all irreducible components W of $V_{\bar{b}} \cap \bar{a}T_i$ of maximal dimension.

The above condition is definable by Corollary 2.5.

Lemma 4.4. *Let $(V_{\bar{z}}(\bar{x}) : \bar{z} \models \theta)$ be a family of code varieties and $\varphi_1(\bar{x}, \bar{z})$ be as above. Then $\varphi_1(\bar{x}, \bar{b})$ is generic in $V_{\bar{b}}$ for $\bar{b} \models \theta$. Moreover, for all $\bar{a} \models \varphi_1(\bar{x}, \bar{b})$ and $B \ni \bar{b}$ the following holds:*

(1) $\delta(\bar{a}/B) \leq 0$.

(2) If $\delta(\bar{a}/B) = 0$, then either $\bar{a} \in \langle B\rangle$ or $\bar{a}$ is generic in $V_{\bar{b}}$ over B.

Proof. The proof of Lemma 4.3 shows that $\varphi_1(\bar{x}, \bar{b})$ is generic in $V_{\bar{b}}$. Suppose now that $\bar{a}$ is neither generic in $V_{\bar{b}}$ nor contained in $\langle B\rangle$. We need to show that $\delta(\bar{a}/B) < 0$. Let W be its locus over $\operatorname{acl}(B)$. As in the previous proof, we consider three cases:

Case 1: $\bar{a} \in \operatorname{acl}(B)$ (equivalently, W is a single point). Then,

$$\delta(\bar{a}/B) = -\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}/B) < 0.$$

Case 2: W is infinite with $\operatorname{cd}(W) = \operatorname{cd}(V_{\bar{b}})$. As above $W \subsetneq V_{\bar{b}}$, so $\delta(\bar{a}/B) \leq \dim(W) - \operatorname{cd}(W) < \dim(V_{\bar{b}}) - \operatorname{cd}(V_{\bar{b}_1}) = 0$.

Case 3: W is infinite with $\mathrm{cd}(W) < \mathrm{cd}(V_{\bar{b}})$. A similar argument as in 4.3 yields the claim. Remark that the conditions on genericity and θ are now part of $\varphi_1(\bar{x}, \bar{b})$. □

Given two formulae $\varphi(\bar{x})$ and $\psi(\bar{x})$ we write $\varphi(\bar{x}) \sim \psi(\bar{x})$ if the Morley rank of their symmetric difference is less than $\mathrm{MR}(\varphi(\bar{x}))$. Therefore $\mathrm{MR}(\varphi) = \mathrm{MR}(\psi)$ and $\sim$ ist symmetric. Let p be a minimal prealgebraic (strong) type, B some set of parameters and $\bar{g} \models p|B$. Then $\bar{g}$ is a $\mathcal{L}_{mult}$-basis of the minimal prealgebraic extension $\langle B\rangle \subseteq \langle B\bar{g}\rangle$. Moreover, $\dim(\bar{g}/B) = k = \frac{n}{2}$ and $\delta(\bar{g}/[\bar{g}T], B) < 0$ for every torus T different from T_0 and T_1. Minimal prealgebraic extensions are preserved under *affine transformations* (which correspond to bases change in $\mathcal{L}_{mult}$). Same holds for strong types as well as degree 1 formulae. Let us explain this carefully: consider $\mathbb{C}^*$ as a $\mathbb{Q}$-vector space modulo torsion. Then, the group $\mathrm{GL_n}(\mathbb{Q})$ acts on the set of strong types *modulo torsion*. If X_1 and X_2 are two definable sets in $(\mathbb{C}^*)^n$ of degree 1 and $T \subseteq (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ is an n-dimensional torus such that $(X_1 \times X_2) \cap T$ projects generically onto both X_1 and X_2, then T induces a *toric correspondence* between X_1 and X_2. The following holds:

Lemma 4.5. *Let $\varphi(\bar{x})$ be minimal prealgebraic.*

- *If there is a toric correspondence between $\varphi(\bar{x})$ and some formula $\psi(\bar{x})$ of Morley degree 1, then $\psi(\bar{x})$ is also minimal prealgebraic.*
- *Let $\bar{m} \in (\mathbb{C}^*)^n$. Then $\varphi(\bar{x} \cdot \bar{m})$ is minimal prealgebraic.* □

Definition 4.6. Let $X \subseteq (\mathbb{C}^*)^n$ be definable set of degree 1. A formula $\varphi(\bar{x}, \bar{z})$ and a torus T *encode* X if there is some $\bar{b}$ such that T induces a toric correspondence between $\varphi(\bar{x}, \bar{b})$ and X. We say that φ *encodes* X if the above correspondence is the identity (i.e. $\varphi(\bar{x}, \bar{b}) \sim X$).

Definition 4.7. A *code* α are integers n_α, k_α and a $\emptyset$-definable formula $\varphi_\alpha(\bar{x}, \bar{z})$ satisfying the following:

(a) The length of $\bar{x}$ is $n_\alpha = 2k_\alpha$.
(b) $\varphi_\alpha(\bar{x}, \bar{b})$ is a subset of $(\mathbb{C}^*)^{n_\alpha}$.
(c) $\varphi_\alpha(\bar{x}, \bar{b})$ is either empty or has Morley rank k_α and Morley degree 1.
(d) If $\varphi_\alpha(\bar{x}, \bar{b}) \neq \emptyset$, then $\varphi_\alpha(\bar{x}, \bar{b})$ is minimal prealgebraic with irreducible Zariski closure $V_\alpha(\bar{x}, \bar{b})$.
(e) Suppose $\varphi_\alpha(\bar{x}, \bar{b}) \neq \emptyset$. Then $\delta(\bar{a}/B) \leq 0$ for every $\bar{b} \in B$ and $\bar{a} \models \varphi_\alpha(\bar{x}, \bar{b})$. Moreover, $\delta(\bar{a}/B) = 0$ if and only if $\bar{a} \in \langle B\rangle$ or $\bar{a}$ is B-generic in $\varphi_\alpha(\bar{x}, \bar{b})$.
(f) $\varphi_\alpha(\bar{x}, \bar{z})$ encodes every multiplicative translate of $\varphi_\alpha(\bar{x}, \bar{b})$.
(g) If $\emptyset \neq \varphi_\alpha(\bar{x}, \bar{b}) \sim \varphi_\alpha(\bar{x}, \bar{b}')$, then $\bar{b} = \bar{b}'$.

If follows from (g) that $\bar{b}$ is the canonical basis of the minimal prealgebraic type determined by $\varphi_\alpha(\bar{x}, \bar{b})$.

Lemma 4.8. *Every minimal prealgebraic definable set X can be encoded by some code α.*

Proof. Let $V_\alpha(\bar{x}, \bar{b})$ be the variety associated to X. Then $V_\alpha(\bar{x}, \bar{b})$ is a code variety. By Lemma 4.3 there is some formula $\theta(\bar{z})$ such that $V_\alpha(\bar{x}, \bar{b}')$ — and all its multiplicative translates — is a code variety for $\bar{b}' \models \theta$. Let now $\varphi_1(\bar{x}, \bar{z}) \subseteq V_\alpha(\bar{x}, \bar{z})$ be as in in Lemma 4.4. Note that for every multiplicative translate $V_\alpha(\bar{x} \cdot \bar{m}, \bar{z})$ the corresponding translate $\varphi_1(\bar{x} \cdot \bar{m}, \bar{z}) \subseteq V_\alpha(\bar{x} \cdot \bar{m}, \bar{z})$ yields the claim in Lemma 4.4. Set now

$$\varphi_\alpha(\bar{x}, \bar{z}\bar{z}') := V(\bar{x} \cdot \bar{z}', \bar{z}) \wedge \varphi_1(\bar{x} \cdot \bar{z}', \bar{z}) \wedge \theta(\bar{z}).$$

Therefore, φ encodes X and satisfies properties (a)-(f).

By definability of $\sim$-equivalence and elimination of imaginaries we may assume that φ also satisfies (g). □

Set now $\theta_\alpha(\bar{z}) := \exists \bar{x}\, \varphi_\alpha(\bar{x}, \bar{z})$.

Lemma 4.9. *Let α and β be codes. Then there is a finite set $G(\alpha, \beta)$ of tori in $(\mathbb{C}^*)^{2n_\alpha}$ such that if $\varphi_\alpha(\bar{x}, \bar{b}) \neq \emptyset$ and T induces a toric correspondence between $\varphi_\alpha(\bar{x}, \bar{b})$ and $\varphi_\beta(\bar{x}, \bar{b}')$, then $T \in G(\alpha, \beta)$.*

Proof. If there is no such toric correspondence between any instances of α and β, then set $G(\alpha, \beta) := \emptyset$. Otherwise, let T, $\bar{b}$ and $\bar{b}'$ as above. Let V_α (*resp.* V_β) be the family of code varieties associated to α (*resp.* β). Moreover, let $\{T_0, \dots, T_\nu\}$ be the finite collection of tori as in 2.2 for $V_\alpha \times V_\beta$. Set $B := \mathrm{acl}(\bar{b}\bar{b}')$. Choose some B–generic point $(\bar{a}, \bar{a}')$ in $(V_\alpha(\bar{x}, \bar{b}) \times V_\beta(\bar{x}', \bar{b}')) \cap T$.

Let $W \subseteq (V_\alpha \times V_\beta) \cap T$ be the locus of $(\bar{a}, \bar{a}')$ over B. Then T is the minimal torus of W, and $\dim(W) = \mathrm{cd}(W) = k_\alpha$.

If we show that $W \subseteq (V_\alpha \times V_\beta)$ is cd-maximal, then T lies in $\{T_0, \dots, T_\nu\}$. Choose now some variety $W \subsetneq W' \subseteq (V_\alpha \times V_\beta)$. We may assume that W' is B–definable. If $(\bar{g}, \bar{g}')$ is B–generic in W', then $\mathrm{cd}(W') = \mathrm{l.dim}_{\mathbb{Q}}(\bar{g}, \bar{g}'/B) - \dim(\bar{g}, \bar{g}'/B)$. Hence,

$$\begin{aligned} \mathrm{cd}(W') &= [\mathrm{l.dim}_{\mathbb{Q}}(\bar{g}/B) - \dim(\bar{g}/B)] + [\mathrm{l.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \dim(\bar{g}'/B\bar{g})] \\ &= \mathrm{cd}(W) + \mathrm{l.dim}_{\mathbb{Q}}(\bar{g}'/B\bar{g}) - \dim(\bar{g}'/B\bar{g}) > \mathrm{cd}(W) - \delta(\bar{g}'/B\bar{g}) \geq \mathrm{cd}(W). \end{aligned}$$

The proper inequality follows from $\dim(\bar{g}'/B\bar{g}) > 0$ because $W \subsetneq W'$. The last inequality follows from the fact that $\bar{g}'$ is B–generic in V_β, so it also realizes $\varphi_\beta(\bar{x}, \bar{b}')$. □

Theorem 4.10. *There exists a collection $\mathcal{C}$ of codes such that every minimal prealgebraic definable set can be encoded by a unique element α in $\mathcal{C}$ and finitely many tori.*

Proof. The collection $\mathcal{C}$ will be obtained as an increasing union of finite sets constructed by recursion. Encode first all minimal prealgebraic subsets of $(\mathbb{C}^*)^n$ for every n. Fix some $n \geq 2$ and list all minimal prealgebraic subsets $(X_i : i < \omega)$ of $(\mathbb{C}^*)^n$ up to isomorphism. Let α_0 encode X_0 as in 4.8. Define $\mathcal{C}_0 = \{\alpha_0\}$.

Suppose by induction that $\mathcal{C}_i$ has been already defined encoding all X_j's with $j \leq i$. If X_{i+1} can be encoded by some element in $\mathcal{C}_i$ and some torus T, then set $\mathcal{C}_{i+1} = \mathcal{C}_i$. Otherwise find some code α_{i+1} and $\bar{b}_0$ as in 4.8 encoding X_{i+1}. Define:

$$\rho(\bar{z}) := \forall \bar{y} \left(\bigwedge_{k=0}^{i} \bigwedge_{T \in G(\alpha_k, \alpha_{i+1})} \neg \chi^T_{\alpha_k, \alpha_{i+1}}(\bar{y}, \bar{z}) \right),$$

where $\chi^T_{\alpha,\beta}(\bar{b}, \bar{b}')$ expresses that T induces a toric correspondence between $\varphi_\alpha(\bar{x}, \bar{b})$ and $\varphi_\beta(\bar{x}, \bar{b}')$ (this is a definable condition).

Now, $\varphi_{\hat{\alpha}}(\bar{x}, \bar{z}) := \varphi_{\alpha_{i+1}}(\bar{x}, \bar{z}) \wedge \rho(\bar{z})$ satisfies properties (a)–(g) in Definition 4.7. We claim that it also satisfies property (f). Let $\bar{m}$ be in $(\mathbb{C}^*)^n$ such that $\varphi_{\hat{\alpha}}(\bar{x} \cdot \bar{m}, \bar{b})$ cannot be encoded by $\varphi_{\hat{\alpha}}$. Equivalently, there are some $k \leq i$ and some torus $T \in G(\alpha_k, \alpha_{i+1})$ such that T induces a toric correspondence between $\varphi_{\alpha_k}(\bar{x}, \bar{b}_1)$ and $\varphi_{\alpha_{i+1}}(\bar{x} \cdot \bar{m}, \bar{b})$. Find $\bar{m}_1 \in (\mathbb{C}^*)^{n_\alpha}$ with $(\bar{m}_1, \bar{m}) \in T$. Then T induces a toric correspondence between $\varphi_{\alpha_k}(\bar{x} \cdot \bar{m}_1^{-1}, \bar{b}_1)$ and $\varphi_{\alpha_{i+1}}(\bar{x}, \bar{b})$. By property (f) the set

$\varphi_{\alpha_k}(\bar{x} \cdot \bar{m}_1^{-1}, \bar{b}_1)$ can be encoded by φ_{α_k}, which yields a contradiction. Hence, set $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{\hat{\alpha}\}$.

Given some minimal prealgebraic definable set X, there is some X_i such that X and X_i are isomorphic. By construction, there exists a unique code $\alpha \in \mathcal{C}$ and finitely many tori which encode X_i (and hence X): finiteness of the set of tori follows from $G(\alpha, \alpha)$ being finite (see Lemma 4.9). □

Definition 4.11. A *good code* is an element of $\mathcal{C}$.

5. Difference sequences

Recall the following result due to M. Ziegler (in a more general setting) in unpublished work [19].

Lemma 5.1. *Let A be an algebraically closed subset. If the tuples $\bar{a}$, $\bar{b}$ and $\bar{a} \cdot \bar{b}$ are pairwise independent over A, then $\operatorname{tp}(\bar{a}/A)$ is the generic type of an A-definable coset of a torus.*

This result is extremely relevant for our purposes due to an observation by Mustafin [14, Proposition 3.1], who noticed that no code variety could be a coset of a torus :

Lemma 5.2. *Let V be a code variety. Then its multiplicative stabiliser is finite. In particular, V is no coset of a torus.*

Proof. For $V = V(\bar{x}, \bar{b})$ as above, choose $T = \operatorname{stab}(V)^0$ the connected component of its multiplicative stabiliser. By construction T is a torus. Choose now generic $\bar{b}$-independent elements $\bar{g}$ in V and $\bar{a}$ in T. By definition, the element $\bar{a} \cdot \bar{g}$ is generic in V over $\bar{b}$. Therefore $\delta(\bar{a}/\bar{g}, \bar{b}) = \delta(\bar{a}/\bar{b}) = \dim(T)$. On the other hand, $\delta(\bar{a} \cdot \bar{g}/\bar{g}, \bar{b}) \leq 0$, since V is a code variety. Since $\bar{a} \cdot \bar{g}$ and $\bar{a}$ are $\bar{g}\bar{b}$-interdefinable (multiplicatively), it follows that $\dim(T) = 0$. □

Definition 5.3. Let $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ a sequence of length $\lambda + 1$. The i^{th} *derivation* ∂_i maps

$$(\bar{e}_0, \ldots, \bar{e}_\lambda)$$

to

$$(\bar{e}_0 \cdot \bar{e}_i^{-1}, \ldots, \bar{e}_{i-1} \cdot \bar{e}_i^{-1}, \bar{e}_i^{-1}, \bar{e}_{i+1} \cdot \bar{e}_i^{-1}, \ldots, \bar{e}_\lambda \cdot \bar{e}_i^{-1}).$$

A sequence obtained after composing the operators $(\partial_i)_{i \leq \lambda}$ finitely many times is a *difference sequence*. If $\nu < \lambda$ and we only consider operators $(\partial_i)_{i \leq \nu}$, then we call the resulting sequence a *ν–difference sequence*.

Note 5.4. For every λ there exist only finitely many different derivations (precisely $(\lambda + 2)!$ many). Moreover, the set of derivations of a given fixed sequence is closed under permutations, since the transposition (ij) equals $\partial_j \circ \partial_i \circ \partial_j$.

Fix for each $\alpha \in \mathcal{C}$ a positive integer m_α such that every $\bar{b} \models \theta_\alpha$ lies in the definable closure of some (*any*) Morley sequence of $\varphi_\alpha(\bar{x}, \bar{b})$ of length m_α.

Theorem 5.5. *For every α in $\mathcal{C}$ and $\lambda \geq m_\alpha$ there is some formula $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda)$ (whose realizations will be called* difference sequences*) satisfying the following:*

(h) *If $\models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_\lambda)$, then $\bar{e}_i \neq \bar{e}_j$ for $i \neq j$.*

(i) *Given $\bar{b} \models \theta_\alpha$ and a Morley sequence $\{\bar{e}_0, \ldots, \bar{e}_\lambda, \bar{f}\}$ for $\varphi_\alpha(\bar{x}, \bar{b})$, then*

$$\models \psi_\alpha(\bar{e}_0 \cdot \bar{f}^{-1}, \ldots, \bar{e}_\lambda \cdot \bar{f}^{-1}).$$

(j) *For any realization* $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ *von* ψ_α *there exists a unique* $\bar{b}$ *with* $\models \varphi_\alpha(\bar{e}_i, \bar{b})$ *for* $i = 0, \ldots, \lambda$*. Moreover,* $\bar{b}$ *lies in the definable closure of any segment of length* m_α *of the* $\bar{e}_i$*'s. Hence,* $\bar{b}$ *is called the* canonical parameter *of the sequence* $\bar{e}_0, \ldots, \bar{e}_\lambda$*.*

(k) *If* $\models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_\lambda)$*, then* $\models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_{\lambda'})$ *for each* $m_\alpha \leq \lambda' < \lambda$*.*

(l) *Let* $i \neq j$ *and* $(\bar{e}_0, \ldots, \bar{e}_\lambda)$ *be a realization of* ψ_α *with canonical parameter* $\bar{b}$ *as in (j). If there is some* T *in* $G(\alpha, \alpha)$ *and* $\bar{e}'_j$ *with* $(\bar{e}_j, \bar{e}'_j) \in T$*, then* $\bar{e}_i \not\perp_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1}$ *in case* $\bar{e}_i$ *is a generic realization of* $\varphi_\alpha(\bar{x}, \bar{b})$*.*

(m) *If* $\models \psi_\alpha(\bar{e}_0, \ldots, \bar{e}_\lambda)$*, then* $\models \psi_\alpha(\partial_i(\bar{e}_0, \ldots, \bar{e}_\lambda))$ *for* $i \in \{0, \ldots, \lambda\}$*.*

Proof. We find $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda)$ inductively in λ. Consider the following type-definable property $\Sigma(\bar{e}_0, \ldots, \bar{e}_\lambda)$:

there exist some $\bar{b}'$ and a Morley sequence $\bar{e}'_0, \ldots, \bar{e}'_\lambda, \bar{f}$ of $\varphi_\alpha(\bar{x}, \bar{b}')$ with $\bar{e}_i = \bar{e}'_i \cdot \bar{f}^{-1}$.

It is easy to see that Σ has properties (h)–(k) and (m). Note that $(\bar{e}_i : i \leq \lambda)$ is a Morley sequence over $\bar{b}'\bar{f}$. In particular, its canonical parameter $\bar{b}$ lies in $\mathrm{dcl}(\bar{b}'\bar{f})$. Let $T \in G(\alpha, \alpha)$ and $(\bar{e}_j, \bar{e}^*_j) \in T$. Then $\bar{e}^*_j \in \mathrm{acl}(\bar{e}_j)$, so $\bar{e}^*_j \perp_{\bar{b}} \bar{e}_i$ for $i \neq j$. If $\bar{e}_i \perp_{\bar{b}} \bar{e}^*_j \cdot \bar{e}_i^{-1}$, then $\bar{e}_i^{-1}$, $\bar{e}^*_j$ and $\bar{e}^*_j \cdot \bar{e}_i^{-1}$ will determine a pairwise $\bar{b}$–independent triple, since

$$\mathrm{MR}(\bar{e}^*_j/\bar{b}, \bar{e}^*_j \cdot \bar{e}_i^{-1}) = \mathrm{MR}(\bar{e}_i^{-1}/\bar{b}, \bar{e}^*_j \cdot \bar{e}_i^{-1}) = \mathrm{MR}(\bar{e}_i^{-1}/\bar{b}) = \mathrm{MR}(\bar{e}_j/\bar{b}) = \mathrm{MR}(\bar{e}^*_j/\bar{b})$$

so $\bar{e}^*_j \perp_{\bar{b}} \bar{e}^*_j \cdot \bar{e}_i^{-1}$. By Lemma 5.1 the type $\mathrm{tp}(\bar{e}_i^{-1}/\bar{b})$ will be the generic type of some torus coset. Likewise for $\mathrm{tp}(\bar{e}_i/\bar{b})$. This contradicts Lemma 5.2 since $\bar{e}_i$ is generic in $\varphi_\alpha(\bar{x}, \bar{b})$. Therefore, property (l) also holds.

Find now by compactness a finite set ψ_0 in Σ which implies (h)-(l). Define

$$\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\lambda) := \bigwedge_{\partial \text{ derivation}} \psi_0(\partial(\bar{x}_0, \ldots, \bar{x}_\lambda)).$$

□

6. Green up!

From now on we will consider an extension $\mathcal{L}^* := \mathcal{L}_{Morley} \cup \{\ddot{\mathrm{U}}\}$, where $\ddot{\mathrm{U}}$ is a new unary predicate which determines the *green coloring* (from the german word *grün*). An $\mathcal{L}^*$-structure is a pair $(A, \ddot{\mathrm{U}}(A))$ consisting of a structure A in $\mathcal{D}$ (i.e. $A = \langle A \rangle \subseteq \mathbb{C}$) and a divisible torsion-free subgroup $\ddot{\mathrm{U}}(A)$ of $A \setminus \{0\}$, that is, a $\mathbb{Q}$-vector space. Given two $\mathcal{L}^*$-structures B and A, we write $B \subseteq A$ in case $B \subset A$ as elements of $\mathcal{D}$ and $\ddot{\mathrm{U}}(A) \cap B = \ddot{\mathrm{U}}(B)$.

The δ-function introduced in Section 3 will be modified accordingly: Given an $\mathcal{L}^*$-structure A finitely generated with respect to $\langle \cdot \rangle$, set

$$\delta(A) := 2 \dim(A) - \mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)).$$

If $B \subseteq A$ and A is f.g. over B, or more generally both $\mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)/\ddot{\mathrm{U}}(B))$ and $\dim(A/B)$ are finite, define

$$\delta(A/B) := 2 \dim(A/B) - \mathrm{l.}\dim_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)/\ddot{\mathrm{U}}(B)). \tag{6.1}$$

This agrees with Poizat's context (*cf.* [16, Section 3]). Hence,*bases*, *generated*, *linearly (in)dependent* refer to the underlying $\mathcal{L}_{mult}$-structure. Similarly, *generic*, *Morley sequence*, *(in)dependent*, *transcendental* or *algebraic* refer to the theory

ACF_0, unless otherwise specified. From now no, $\mathrm{acl}(M)$ denotes the field theoretical algebraic closure of M.

Note 6.1. Let $B \subseteq A$ as above. If A has a green basis over B, that is $A = \langle \ddot{\mathrm{U}}(A)B\rangle$, then the previous definition agrees with the definition of $\delta(A/B)$ in Section 3. In particular, property (e) in Definition 4.7 holds for green realizations of a code instance $\varphi_\alpha(\bar{x}, \bar{b})$.

Given $B \subseteq A$, we say that B is *strong* in A, if $\delta(A'/B) \geq 0$ for every $A' = \langle A'\rangle \subset A$ f.g over B. We denote it by $B \leq A$. If there is some $\bar{b} \in B$ generating B, then we write $\bar{b} \leq A$ in case $B \leq A$. Similarly, set $\delta(\bar{a}/\bar{b}) := \delta(\langle \bar{a}\bar{b}\rangle/\langle \bar{b}\rangle)$.

Lemma 6.2. *Suppose all structures can be embedded into a common $\mathcal{L}^*$-structure M. Then:*

(1) For $B \subseteq C \subseteq A$, we have $\delta(A/B) = \delta(C/B) + \delta(A/C)$.
(2) $\delta(\langle AB\rangle/B) \leq \delta(A/A \cap B)$. (Submodularity)
(3) If $C \leq M$ and $C' \leq M$, then $C \cap C' \leq M$.
(4) For every $A \subseteq M$ there exists a unique $A \subseteq C = \langle C\rangle \leq M$ minimal such. We call such a set the strong closure *of A (in M) and denote it by $\mathrm{cl}_M(A)$.*
(5) If $(A_i)_{i<\alpha}$ is an increasing sequence with $A_i \leq K$ for all i, then $\bigcup_i A_i \leq M$.

Now consider the class of $\mathcal{L}^*$-Structures $\mathcal{K} := \{M \mid \emptyset \leq M\}$. Unlike in [16] we are not interested in $\mathcal{L}^*$-structures whose underlying $\mathcal{L}_{Ring}$-structure is an algebraically closed field but mere expansions of structures in $\mathcal{D}$ with hereditarily non-negative predimension function δ.

Assumption 6.3. From now on, δ will be as in in (6.1). A realization of a code or a difference sequence will consist exclusively of green elements, unless otherwise specified. Likewise, a *minimal prealgebraic* extension $M \leq N$ in $\mathcal{K}$ is a minimal prealgebraic extension of structures N/M in $\mathcal{D}$ such that N has a green basis over M.

Given a strong extension $B \leq A$ in $\mathcal{K}$ with $\mathrm{l.dim}_{\mathbb{Q}}(A/B) < \infty$, we can find a decomposition $B = A_0 \leq A_1 \leq \ldots \leq A_{n-1} \leq A_n = A$ such that A_{i+1}/A_i is minimal strong for all $i < n$. Note that a strong extension $M \leq N$ in $\mathcal{K}$ ($M \subsetneq N$) is *minimal (strong)* if there is no $M' = \langle M'\rangle$ with $M < M' \leq N$. The following result is easy to prove.

Lemma 6.4. *Let $B \leq A$ be a minimal extension. One of the following cases holds:*

(1) algebraic: *$\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$ and $A = \langle Ba\rangle$ for some $a \in \mathrm{acl}(B) \setminus B$. Then, $\delta(A/B) = 0$.*
(2) white generic: *$\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$ and $A = \langle Ba\rangle$ for some element a transcendental over B. Then, $\delta(A/B) = 2$.*
(3) green generic: *A contains a basis consisting of a green singleton a over B. Moreover, a is transcendental over B and $\delta(A/B) = 1$.*
(4) minimal prealgebraic: *$B \leq A$ is minimal prealgebraic as in 6.3, that is, A contains a green basis $\bar{a}$ over B such that the (strong) type of $\bar{a}$ over B is minimal prealgebraic. In this case $\delta(A/B) = 0$.*

The class $\mathcal{K}$ can be easily axiomatized as shown in [16]:

Theorem 6.5. *Let $(M, \ddot{\mathrm{U}}(M))$ be an $\mathcal{L}^*$-expansion of an algebraically closed field of characteristic 0. Then $M \in \mathcal{K}$ if and only if the following (definable) conditions hold:*

(1) $\ddot{\mathrm{U}}(M)$ *is a trosion-free divisible multiplicative subgroup of* M.
(2) $\ddot{\mathrm{U}}(M)$ *has no non-trivial algebraic points.*
(3) *For every* $\emptyset$*-definable variety* $V(\bar{x}) \subseteq (\mathbb{C}^*)^{2n+1}$ *of dimension* n *with associated tori* $\{T_0, \ldots, T_r\}$ *as in 2.2, then* $\bar{a} \in T_i$ *for some* $i > 0$ *for all* $\bar{a} \in V \cap \ddot{\mathrm{U}}(M)$.

Note that (2) follows from (1) and (3).

The class $\mathcal{K}$ has the amalgamation property with respect to strong embeddings. Moreover, it has the JEP and contains only countably many f.g. structures up to isomorphism. Hence, the Fraïssé-Hrushovski limit M_ω of the subcollection of f.g. structures in $(\mathcal{K}, \leq)$ exists. We call M_ω the *generic model* of $\mathcal{K}$. Let T_ω be the $\mathcal{L}^*$-theory of M_ω. Recall the following result from [16]:

Theorem 6.6. *The generic model* M_ω *is* ω*-saturated. Its theory* T_ω *is* ω*-stable of Morley rank* $\omega \cdot 2$. *Moreover,* $\ddot{\mathrm{U}}(M_\omega)$ *has Morley rank* ω.

Note 6.7. Let $V_{\bar{z}}(\bar{x}) : \bar{z} \models \theta)$ be a family of code varieties and $\varphi_1(\bar{x}, \bar{z})$ as in Lemma 4.4. Then, for all $\bar{b} \models \theta$, the formula $\varphi_1(\bar{x}, \bar{b}) \wedge \bigwedge_{i=1}^n \ddot{\mathrm{U}}(x_i)$ defines a strongly minimal set in the theory T_ω.

7. Green counts

This section contains the main result of a combinatorial flavour, which will be extremely useful in order to show that the generic model M_ω can be collapsed into a finite rank one.

Definition 7.1. Let A and M be elements in $\mathcal{K}$ with a common strong substructure B. An $\mathcal{L}^*$-structure M' in $\mathcal{K}$ is an *amalgam* of A and M over B if A and M are strongly embedded in M' over B and $M' = \langle M, A \rangle$ (after identification of A and M with their images under their respective embeddings). In case M and A (or rather, their images in M') are algebraically independent over B and $M \cap A = B$, then M' is a *free amalgam.*

Following [16] we obtain the following:

Lemma 7.2. *Given* M, A *and* B *in* $\mathcal{K}$ *with* $B \leq A$ *and* $B \leq M$. *Then there is an amalgam* M' *in* $\mathcal{K}$ *of* A *and* M *over* B *such that* A *and* M *are algebraically independent over* B. *If* B *is algebraically closed in* A *or in* M, *then the amalgam can chosen to be a free amalgam.*

The following Lemma yields a lower bound for the length of a difference sequence for a good code in order to recover a Morley segment inside the sequence over a strong subset of parameters.

Lemma 7.3. *For every code* α *and every natural number* n *there exists a positive integer* $\lambda_\alpha(n) = \lambda \geq 0$ *such that given any strong extension* $M \leq N$ *in* $\mathcal{K}$ *and a difference sequence* $(\bar{e}_0, \ldots, \bar{e}_\mu)$ *for* α *in* N *with canonical parameter* $\bar{b}$, *we have that if* $\mu \geq \lambda$ *then either the canonical parameter of a* λ*-derived sequence of* $(\bar{e}_0, \ldots, \bar{e}_\mu)$ *lies in* $\mathrm{acl}(M)$, *or the sequence* $(\bar{e}_0, \ldots, \bar{e}_\mu)$ *contains a Morley subsequence for* $\varphi_\alpha(\bar{x}, \bar{b})$ *over* M *of length* n.

Proof. Given $(\bar{e}_0,\dots,\bar{e}_\mu)$ as above such that first part of the statement does not hold, define:

$$\begin{aligned} X_1 &= \{i\in[m_\alpha,\mu] : \bar{e}_i \text{ generic over } M\cup\{\bar{e}_0,\dots,\bar{e}_{i-1}\}\},\\ X_2 &= \{i\in[m_\alpha,\mu] : \bar{e}_i\subseteq\langle M\cup\{\bar{e}_0,\dots,\bar{e}_{i-1}\}\rangle\},\\ X_3 &= [m_\alpha,\mu]\setminus(X_1\cup X_2).\end{aligned}$$

After possible permutation of the set of indices, we may assume that $X_1<X_3<X_2$ (note that we may have indices of X_2 go to X_3 and indices of X_3 land in X_1). Since $\bar{b}\in\operatorname{dcl}(\bar{e}_0,\dots,\bar{e}_{m_\alpha-1})$, then by Property (e)

$$\begin{aligned} \delta(\bar{e}_i/M,\bar{e}_0,\dots,\bar{e}_{i-1}) &\leq -1 \quad\text{for } i\in X_3 \text{ and}\\ \delta(\bar{e}_i/M,\bar{e}_0,\dots,\bar{e}_{i-1}) &= 0 \quad\text{for } i\in X_1\cup X_2.\end{aligned}$$

It follows from $M\leq N$ that

$$\begin{aligned} 0\leq\delta(\bar{e}_0,\dots,\bar{e}_\mu/M) &\leq \delta(\bar{e}_0,\dots,\bar{e}_{m_\alpha-1}/M)+\sum_{i=m_\alpha}^{\mu}\delta(\bar{e}_i/M,\bar{e}_0,\dots,\bar{e}_{i-1})\\ &\leq m_\alpha n_\alpha+(-1)|X_3|\ ,\end{aligned}$$

therefore $|X_3|\leq m_\alpha n_\alpha$.

Let now $r=m_\alpha+|X_1|+|X_3|$, and $s=r(n_\alpha+1)$. Take $I\subseteq\{\bar{e}_r,\dots,\bar{e}_\mu\}$ of cardinality $|I|=rn_\alpha+1$. To simplify the notation, assume that $I=\{r,\dots,s\}$. Choose varieties $W_0\subset V_0,\dots,W_t\subset V_t$ (with the V_i's irreducible) such that $\psi_\alpha(\bar{x}_0,\dots,\bar{x}_s)$ equals $\bigcup_{i\leq t}(V_i\setminus W_i)$. Let $T_0,\dots,T_\ell$ be the associated tori to the V_i's as in 2.2. The point $(\bar{e}_0,\dots,\bar{e}_s)$ lies in some $V_{i_0}\setminus W_{i_0}$ for some $i_0\leq t$. Let W be its locus over $\operatorname{acl}(M)$. Choose $W\subseteq W'\subseteq V_{i_0}$ maximal such that $\operatorname{cd}(W')\leq\operatorname{cd}(W)$. By construction W' is cd-maximal, so there is some $j\in\{0,\dots\ell\}$ such that T_j is its minimal torus. Fix some $\bar{m}$ in $W'\subseteq\bar{m}T_j\cap V_{i_0}$. We may assume that $\bar{m}\in\operatorname{acl}(M)$, since $W\subseteq\bar{m}T_j$ by $\operatorname{acl}(M)$-definability of W'. Choose $(\bar{a}_0,\dots,\bar{a}_s)$ a generic point of W' over $\operatorname{acl}(M)$ and paint it green. Then, the point lies in $V_{i_0}\setminus W_{i_0}$, since $(\bar{e}_0,\dots,\bar{e}_s)$ was an specialization of $(\bar{a}_0,\dots,\bar{a}_s)$ and $V_{i_0}\setminus W_{i_0}$ is Zariski open in its closure. Therefore $\psi_\alpha(\bar{a}_0,\dots,\bar{a}_s)$ holds. Note that

$$\begin{aligned} r\cdot n_\alpha &\geq \operatorname{l.dim}_{\mathbb{Q}}(\bar{e}_0,\dots,\bar{e}_{r-1}/M)=\operatorname{l.dim}_{\mathbb{Q}}(\bar{e}_0,\dots,\bar{e}_s/M)\geq\operatorname{cd}(W)\geq\operatorname{cd}(W')\\ &=\sum_{i\leq s}\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})-\dim(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})\\ &\geq\sum_{r\leq i\leq s}\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})-\dim(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1}).\end{aligned}$$

Property (e) yields that $\delta(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})\leq 0$ for $i\geq r\geq m_\alpha$, that is,

$$2\dim(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})\leq\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1}).$$

Hence, if $\bar{a}_i\notin\langle\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1}\rangle$ then

$$\operatorname{l.dim}_{\mathbb{Q}}(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})-\dim(\bar{a}_i/\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{i-1})\geq 1.$$

Therefore, there is some $t\in\{r,\dots,s\}$ with $\bar{a}_t\in\langle\operatorname{acl}(M),\bar{a}_0,\dots,\bar{a}_{t-1}\rangle$. The linear dependence will be determined by the coset $\bar{m}T_j$. So $\bar{m}T_j$ also determines that $\bar{e}_t\in\langle\operatorname{acl}(M),\bar{e}_0,\dots,\bar{e}_{t-1}\rangle$.

Consider now all possible pairs (t,j) as above. This determines a $(rn_\alpha+1)(\ell+1)$-coloring of all $(rn_\alpha+1)$-subsets of $\{r,\dots,\mu\}$. By (finite) Ramsey's theorem, there is some number μ_0, such that for $\mu\geq\mu_0$ there is a monochromatic subset $I\subseteq$

$\{r, \dots, \mu\}$ of cardinality $|I| \geq m_\alpha + rn_\alpha + 1$. Equivalently, there is some $t \in \{r, \dots, s\}$ and some $j \leq \ell$ with

$$\bar{e}_{i_t} \in \langle \operatorname{acl}(M), \bar{e}_0, \dots, \bar{e}_{r-1}, \bar{e}_{i_r}, \bar{e}_{i_{r+1}}, \dots, \bar{e}_{i_{t-1}} \rangle,$$

for all $i_r < \cdots < i_s$ in I. Moreover, the linear dependence comes from some $\bar{m}T_j$ with $\bar{m}$ in $\operatorname{acl}(M)$ (note that the tuple $\bar{m} \in \operatorname{acl}(M)$ may change). Let γ_i be the $(t+i)^{\text{th}}$ element in I. For $i > 0$ we have that $\bar{e}_{\gamma_i}\bar{e}_{\gamma_0}^{-1}$ lies in $\operatorname{acl}(M)$. Hence, the canonical parameter of the difference sequence lies in

$$\operatorname{acl}(\partial_{\gamma_0}(\bar{e}_0, \dots, \bar{e}_\mu)) \subseteq \operatorname{acl}(M),$$

which contradicts our assumption.

Hence, there is an upper bound for μ depending on r, hence a lower bound for X_1 depending on μ. □

8. Prealgebraicity go home!

Choose now finite-to-one functions $\mu^*, \mu : \mathcal{C} \to \mathbb{N}$ such that:

$$\begin{aligned} \mu^*(\alpha) &\geq n_\alpha k_\alpha + 1, \\ \mu^*(\alpha) &\geq \lambda_\alpha(m_\alpha + 1), \\ \mu(\alpha) &\geq \lambda_\alpha(\mu^*(\alpha)). \end{aligned}$$

where λ_α is the function obtained in Lemma 7.3.

Definition 8.1. Let $\mathcal{K}^\mu$ be the subcollection of elements M in $\mathcal{K}$ such that no good code α has a (green) difference sequence in M of length at least $\mu(\alpha) + 1$.

The class $\mathcal{K}^\mu$ is universally axiomatizable relative to ACF_0. We want now to obtain a theory T^μ whose models lie in $\mathcal{K}^\mu$. Moreover, green prealgebraic extensions of strong subsets will become algebraic in T^μ. In order to ensure that T^μ is complete, we will impose that every good code attains the maximal number of realizations and moreover describe all this in an elementary way.

Lemma 8.2. *Let $M \in \mathcal{K}^\mu$ and $M' \in \mathcal{K}$ be a minimal prealgebraic extension of M not in $\mathcal{K}^\mu$. Given a good code α and a difference sequence $(\bar{e}_0, \dots, \bar{e}_{\mu(\alpha)})$ in M' with canonical parameter $\bar{b}$ in* $\operatorname{acl}(M)$*, then there exists some i such that all $\bar{e}_j$'s with $i \neq j$ lie in M and $\bar{e}_i$ is an M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$ which generates M' over M.*

Proof. We may assume that M is algebraically close, for otherwise we may define an $\mathcal{L}^*$-structure on $\operatorname{acl}(M)$,by setting $\ddot{\mathrm{U}}(\operatorname{acl}(M)) = \ddot{\mathrm{U}}(M)$; minimality of the extension M'/M yields that M is algebraically closed in M' so we could replace M' by $\langle M' \operatorname{acl}(M) \rangle$. Since $M \leq M'$, then it follows from property (e) that for each j either $\bar{e}_j$ lies in M or is a generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$ over M. Since $M \in K^\mu$, there must be some generic $\bar{e}_i$. By minimality of the extension, it follows that $M' = \langle M\bar{e}_i \rangle$. Suppose there is another M-generic realization $\bar{e}_j$ different from $\bar{e}_i$. Then $M' = \langle M\bar{e}_i \rangle = \langle M\bar{e}_j \rangle$, so there is some tuple $\bar{m}$ in M and a toric correspondence induced by some $T \in G(\alpha, \alpha)$ with

$$(\bar{e}_i \cdot \bar{m}, \bar{e}_j) \in T.$$

Let $\bar{e}'_j := \bar{e}_i \cdot \bar{m}$. In particular, $\bar{e}'_j \cdot \bar{e}_i^{-1} \in M$. Since $\bar{e}_i$ is M-generic, then

$$\bar{e}_i \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}} \bar{e}'_j \cdot \bar{e}_i^{-1},$$

which contradicts property (l) of a difference sequence. □

Corollary 8.3. *Let $M \in \mathcal{K}^\mu$ and $M' \in \mathcal{K}$ be a minimal extension of M. If $\operatorname{l.dim}_{\mathbb{Q}}(M'/M) = 1$, then M' lies also in $\mathcal{K}^\mu$. Otherwise, M'/M is minimal prealgebraic, and M' does not lie in $\mathcal{K}^\mu$ if and only if there is some good $\alpha \in \mathcal{C}$ and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' such that one of the following holds:*

a) *$\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ lie in M and $\langle M, \bar{e}_{\mu(\alpha)}\rangle = M'$. Moreover, α is the unique good code describing the extension $M' \geq M$.*

b) *There is some subsequence of length $\mu^*(\alpha)$ which is a Morley sequence for $\varphi_\alpha(\bar{x}, \bar{b})$ over $M\bar{b}$, where $\bar{b}$ is the canonical parameter of $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$.*

Proof. Consider first the case $\operatorname{l.dim}_{\mathbb{Q}}(M'/M) = 1$. If $\ddot{\mathrm{U}}(M') = \ddot{\mathrm{U}}(M)$, then there are no new green difference sequences in M', so $M' \in \mathcal{K}^\mu$. Otherwise, $\operatorname{l.dim}_{\mathbb{Q}}(\ddot{\mathrm{U}}(M')/M) = 1$ and $M' = \langle \ddot{\mathrm{U}}(M'), M\rangle$ (the green generic case). Suppose M' is not in K^μ, then there is a good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' witnessing this fact. Let $\bar{b}$ be the canonical parameter of some derived sequence and $\bar{e}$ be a generic element over $M\bar{b}$. Then

$$\operatorname{l.dim}_{\mathbb{Q}}(M'/M) \geq \operatorname{l.dim}_{\mathbb{Q}}(\bar{e}/M\bar{b}) \geq 2,$$

which contradicts our assumption. By Lemma 8.2 there is no such derived sequence. However, Lemma 7.3 yields a contradiction since $\mu(\alpha) \geq \lambda_\alpha(\mu^*(\alpha))$ and $\mu^*(\alpha) \geq 1$.

Let now M' be a minimal prealgebraic extension of M. If a) or b) hold, then clearly M' does not belong to K^μ. Conversely, if M' is not in K^μ, then there is some good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ for α in M' witnessing this fact. Let $\bar{b}$ be its canonical parameter. If we may choose the difference sequence such that $\bar{b}$ lies in $\operatorname{acl}(M)$, then case a) holds by Lemma 8.2. Otherwise no difference sequence has canonical parameter in $\operatorname{acl}(M)$, which yields case b) because $\mu(\alpha) \geq \lambda_\alpha(\mu^*(\alpha))$ by Lemma 7.3.

In order to show that α is uniquely determined, consider another good code α' different from α with $M' = \langle M, \bar{e}_{\mu(\alpha')}\rangle$. Then $n_\alpha = n_{\alpha'} = \operatorname{l.dim}_{\mathbb{Q}}(M'/M)$ and the locus of $(\bar{e}_{\mu(\alpha)}, \bar{e}_{\mu(\alpha')})$ over M determines a coset of a torus in $G(\alpha, \alpha')$. Since both α and α' are good, $G(\alpha, \alpha') = \emptyset$ by construction, obtaining hence the desired contradiction. □

Corollary 8.4. *Given a good code α there is a $\forall\exists$-sentence χ_α such that every structure M in $\mathcal{K}^\mu$ satisfies χ_α if and only if it contains no minimal prealgebraic extension in $\mathcal{K}^\mu$ given by α.*

Proof. Let $\alpha \in \mathcal{C}$, M in $\mathcal{K}^\mu$ and $\bar{b} \in M$ such that a generic realization $\bar{a}$ of $\varphi_\alpha(\bar{x}, \bar{b})$ generates a minimal prealgebraic extension $M[\bar{a}] := \langle M\bar{a}\rangle$ over M. If $M[\bar{a}]$ does not belong to $\mathcal{K}^\mu$, either case a) or b) in 8.3 hold.

a) is equivalent to the existence of some good code α' and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha')})$ in M', whose first $\mu(\alpha')$ many elements (and hence its canonical parameter) lie in M and $M[\bar{a}] = \langle M\bar{e}_{\mu(\alpha')}\rangle$. By uniqueness, $\alpha = \alpha'$. Since $M \leq M[\bar{a}]$, it follows that $\bar{e}_{\mu(\alpha)}$ is M-generic. Therefore, $\bar{a}$ can be mapped to $\bar{e}_{\mu(\alpha)}$ over M by some $\mathbb{Q}$–basis change. Equivalently, there is some green tuple $\bar{m} \in M$ and some torus $T \in G(\alpha, \alpha)$ such that T induces a toric correspondence between $\psi_\alpha(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}, \bar{x})$ and $\varphi_\alpha(\bar{x} \cdot \bar{m}, \bar{b})$. Finiteness of $G(\alpha, \alpha)$ allows us to express this fact by an existential formula over $\bar{b}$.

b) implies that there is some good code β and a difference sequence $(\bar{e}_0, \dots, \bar{e}_{\mu(\beta)})$ in $M[\bar{a}]$ with $\mu^*(\beta)$ many M-linearly independent elements. We need only consider finitely many such $\beta's$ since

$$n_\beta \mu^*(\beta) \leq \mathrm{l.dim}_{\mathbb{Q}}(M[\bar{a}]/M) = n_\alpha.$$

Assume first the following

Assumption: ψ_β equals $V_1 \setminus W_1$, where V_1 is a irreducible variety and $W_1 \subsetneq V_1$ is a proper subvariety, both $\mathrm{acl}(\emptyset)$-definable.

Let $V_0 = V_\alpha(\bar{x}, \bar{b})$ be the corresponding code variety for $\varphi_\alpha(\bar{x}, \bar{b})$, that is, the locus of $\bar{a}$ over $\mathrm{acl}(M)$. Set $V = V_0 \times V_1$ and let $\{T_0, \dots, T_r\}$ be the tori associated to V as in 2.2. Take $W \subseteq V$ to be the locus of $(\bar{a}, \bar{e}_0, \dots, \bar{e}_{\mu(\beta)})$ over $\mathrm{acl}(M)$. Note that $\mathrm{cd}(W) = \mathrm{cd}(V_0) = k_\alpha$ by 3.2. Since W projects generically onto V_0, then $\mathrm{cd}(W') \geq \mathrm{cd}(W)$ for all $W' \supseteq W$. Let now T be the minimal torus of W and $\bar{m} \in M$ such that $W \subseteq \bar{m}T$. The coset $\bar{m}T$ contains hence the green tuple $(\bar{a}, \bar{e}_0, \dots, \bar{e}_{\mu(\beta)})$.

We call a torus coset $\bar{c}T$ *gay* (as in colorful, let the distinction be made) if it contains a green tuple . In case T is given by the equations $\prod x_i^{\lambda_{ij}} = 1$ $(j = 1, \dots, d)$, it is easy to see that $\bar{c}T$ is gay if and only if $c'_j := \prod c_i^{\lambda_{ij}}$ is green for $j = 1, \dots, d$. If $\bar{c}T$ is gay and $T \subseteq T'$ (that is, T' lies on top of T), then $\bar{c}T'$ is also gay.

Choose now some M-definable W' maximal such that $\mathrm{cd}(W') = \mathrm{cd}(W)$ containing W. Hence, W' is cd-maximal and its minimal torus equals some T_i. Moreover, $W' \subseteq \bar{m}T_i$. Since $\bar{m}T \subseteq \bar{m}T_i$ and $\bar{m}T$ is gay, so is $\bar{m}T_i$. Let $(\bar{a}^*, \bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$ be generic in W' over M. We may assume that $\bar{a}^* = \bar{a}$ since they have the same M-type. It follows from $\mathrm{cd}(W') = \mathrm{cd}(W) = \mathrm{cd}(V_0)$ that

$$\begin{aligned}\mathrm{l.dim}_{\mathbb{Q}}(\bar{a}, \bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)}/\mathrm{acl}(M)) &- \dim(\bar{a}, \bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)}/\mathrm{acl}(M)) \\ &= \mathrm{l.dim}_{\mathbb{Q}}(\bar{a}/\mathrm{acl}(M)) - \dim(\bar{a}/\mathrm{acl}(M))\end{aligned}$$

so

$$\mathrm{l.dim}_{\mathbb{Q}}(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)}/M\bar{a}) \dim(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)}/M\bar{a}) =: \ell.$$

Choose now an $\mathcal{L}_{mult}$-basis $f_0, \dots, f_{\ell-1}$ in $(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$ over $M\bar{a}$. The elements $(f_0, \dots, f_{\ell-1})$ are hence algebraically independent over $M\bar{a}$. Gayness of $\bar{m}T_i$ yields a structure N in $\mathcal{K}$ if we paint $(\bar{a}^*, \bar{e}^*_{\leq\mu(\beta)})$ green (after closing it under $\langle\cdot\rangle$). Note that

$$N = \langle M\bar{a}\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)}\rangle = \langle M[\bar{a}]f_0, \dots, f_{\ell-1}\rangle,$$

where $f_0, \dots, f_{\ell-1}$ is a tuple of green independent generic elements. Set $F_i := \langle M\bar{a}f_0, \dots, f_{i-1}\rangle$ and observe that $F_i \leq F_{i+1}$ gives a tower of green generic extensions for $0 \leq i \leq \ell - 1$. By Collorary 8.3 (repeatedly) we have that:

(*) $M[\bar{a}] \in \mathcal{K}^\mu$ if and only if $N \in \mathcal{K}^\mu$.

Now, $(\bar{e}_0, \dots, \bar{e}_{\mu(\beta)})$ is a specialization of $(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$, both lying in F. By assumption ψ_β is Zariski open in F, so $\models \psi_\beta(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$ since $(\bar{e}_0, \dots, \bar{e}_{\mu(\beta)})$ realizes ψ_β. Therefore, the existence of a green difference sequence for β (of length $\mu(\beta)+1$) implies the existence of another one in $N = M[\bar{a}][\bar{f}]$, which may be obtained by only finitely many possibilities. Conversely, it suffices to ensure the existence of $(\bar{e}^*, \dots, \bar{e}^*_{\mu(\beta)}) \in N$ to conclude that $M[\bar{a}] \notin \mathcal{K}^\mu$ by (*). Consider the following definable conditions:

There is a tuple $\bar{m} \in M$ and an irreducible component W' of $V \cap \bar{m}T_i$ (where $V = V_0 \times V_1$ and $V_0 = V_\alpha(\bar{x}, \bar{b})$ is as above) such that:

(1) The coset $\bar{m}T_i$ is gay.
(2) W' projects generically onto V_0.
(3) $\mathrm{cd}(W') = \mathrm{cd}(V_0)$.
(4) $\models \psi_\beta(\bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$ for generic $(\bar{a}^*, \bar{e}^*_0, \dots, \bar{e}^*_{\mu(\beta)})$ in W'.

Therefore, we obtain an existential sentence over $\bar{b}$ for each T_i. The disjunction of all these formulae yields the desired sentence.

For the general case, decompose ψ_β into a finite union of locally closed sets $V_i \setminus W_i$ (for $1 \leq i \leq t$). We proceed as above for each i and form the disjunction of all the sentences so obtained. □

9. Fraïssé à la verte

This section shows that $\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings. Hence, we obtain a *rich* field as in [15]. Work done in previous sections yields now the following key result:

Lemma 9.1. *Let A, B and M be structures in $\mathcal{K}^\mu$, where B is a common strong substructure of both A and M. Let M' be their free amalgam over B and consider a difference sequence $(\bar{e}_0, \dots, \bar{e}_{\mu(\alpha)})$ in M' for some good code α. Then there is some derived sequence whose canonical parameter lies either in* $\mathrm{acl}(M)$ *or in* $\mathrm{acl}(A)$*.*

Proof. Suppose the statement does not hold. Then, find $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_{m_\alpha}$ of length $m_\alpha + 1$ which is a Morley sequence over both M and A after applying Lemma 7.3 twice by choice of $\mu(\alpha)$ and $\mu^*(\alpha)$. Let $E = \{\bar{e}_0, \dots, \bar{e}_{m_{\alpha-1}}\}$. The canonical parameter $\bar{b}$ lies in $\mathrm{dcl}(E)$ and

$$\bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{b}} ME \qquad \text{und} \qquad \bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{b}} AE\,.$$

Decompose each tuple in E as a product of a green tuple in M and one in A and define E_M (*resp.* E_A) to be the collection of these factors in M (*resp.* A). Set $E' = E_M \cup E_A$. Then $\bar{b} \in \mathrm{dcl}(E')$ and by interdefinability of E und E' over M (*resp.* over A), we conclude that

$$\bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{b}} ME' \qquad \text{und} \qquad \bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{b}} AE'\,,$$

and

$$\bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{BE'} M \qquad \text{und} \qquad \bar{e}_{m_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}_{BE'} A\,.$$

Let $\bar{e}_{m_\alpha} = \bar{m} \cdot \bar{a}$ with $\bar{m}$ in M and $\bar{a}$ in A. Since $M \mathop{\smile\!\!\!\!\!\!\;|}_B A$ then $M \mathop{\smile\!\!\!\!\!\!\;|}_{BE'} A$, so $\{\bar{e}_{m_\alpha}, \bar{m}, \bar{a}\}$ is a pairwise BE'-independent triple. This contradicts Lemma 5.2 by Lemma 5.1, since $\mathrm{stp}(\bar{e}_{m_\alpha}/BE')$ will be the generic type of a torus coset. □

An embedding B in A is *strong* if the image of B in A is a strong substructure.

Theorem 9.2. *$\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings.*

Proof. let $B \leq M$ and $B \leq A$ be structures in $\mathcal{K}^\mu$. We need to find a strong extension M' of M in $\mathcal{K}^\mu$ with $B \leq A' \leq M'$, where A and A' are B-isomorphic. Decomposing both $B \leq A$ and $B \leq M$ in minimal extensions, we may reduce it to the case where both A and M are minimal extensions of B. If any of them is algebraic, add the corresponding elements to the $\langle\cdot\rangle$-closure (we obtain a new structure in $\mathcal{K}^\mu$ since there are no new green points).

Otherwise, we may consider the free amalgam M' of M and A over B by Lemma 7.2. If $M' \in K^{\mu}$, we are done. Otherwise, we need only show that M and A are B-isomorphic. It follows from Corollary 8.3 that both M and A are minimal prealgebraic over B. Note that only the first case in Lemma 9.1 may occur, so there is a good code α and a difference sequence $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$ in M' with canonical parameter $\bar{b}$. By symmetry we may assume that $\bar{b}$ lies in $\mathrm{acl}(M)$. After possible permutations we may assume by Lemma 8.2 that $\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)-1}$ lie in M and $\bar{e}_{\mu(\alpha)}$ is an M-generic realization of α.

Case 1. There is some $(\mu(\alpha)-1)$-derived difference sequence with canonical parameter in $\mathrm{acl}(B)$.

Work hence with the above sequence, which we will still denote by $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$. It usffices to show that $\bar{e}_{\mu(\alpha)}$ lies in A. Otherwise, $\bar{e}_{\mu(\alpha)}$ is generic in $\varphi_\alpha(\bar{x}, \bar{b})$ over A and M, so independent from A and from M over B. Find two green tuples $\bar{a} \in A$ and $\bar{m} \in M$ with $\bar{e}_{\mu(\alpha)} = \bar{m} \cdot \bar{a}$. Observe that $\bar{e}_{\mu(\alpha)}$, $\bar{a}$ and $\bar{m}$ are pairwise B-independent, whihc contradicts Lemma 5.2 by Lemma 5.1. Minimal prealgebraicity of A over B implies that $A = \langle B, \bar{e}_{\mu(\alpha)} \rangle$. Since $A \in K^{\mu}$, there is some $\bar{e}_i$ in $M \setminus B$. Since $B \leq M$, then $\bar{e}_i$ is B-generic by property (e). Hence, the map $\bar{e}_i \mapsto \bar{e}_{\mu(\alpha)}$ induces a B-isomorphism between A and M.

Case 2. No $(\mu(\alpha)-1)$-derivation has canonical parameter in $\mathrm{acl}(B)$.

As above decompose $\bar{e}_{\mu(\alpha)} = \bar{m} \cdot \bar{a}$ with $\bar{m} \in M$ and $\bar{a} \in A$ both green tuples. Since $\bar{e}_{\mu(\alpha)}$ is M-generic, then $0 = \delta(\bar{e}_{\mu(\alpha)}/M) = \delta(\bar{a}/M)$, so $\bar{a}$ generates A over B. Apply now Lemma 7.3 to $B \leq M'$ and $(\bar{e}_0, \ldots, \bar{e}_{\mu(\alpha)})$. There is some Morley segment of length $\mu^*(\alpha)$ over $B\bar{b}$.

Since

$$\mu^*(\alpha) \geq n_\alpha k_\alpha + 1 > n_\alpha \geq MR\ (\bar{m}/B\bar{b}),$$

there is some $\bar{e}_i$ in M with $\bar{m} \mathop{\smile\!\!\!\!\!\!|}_{B\bar{b}} \bar{e}_i$. In particular, $\bar{e}_{\mu(\alpha)}$ and $\bar{e}_i$ have the same type over $B\bar{b}\bar{m}$, and so do $\bar{a} = \bar{m} \cdot \bar{e}_{\mu(\alpha)}^{-1}$ and $\bar{m} \cdot \bar{e}_i^{-1}$. By minimality, $\bar{a} \mapsto \bar{m} \cdot \bar{e}_i^{-1}$ induces a B-isomorphism between A and M. □

Using notation developed by Poizat [15] we say that an $\mathcal{L}^*$-structure M in $\mathcal{K}^{\mu}$ is *rich* if for every f.g. $B \leq M$ and every f.g. strong extension $B \leq A$ in $\mathcal{K}^{\mu}$ there is some strong substructure $A' \leq M$ with $A \simeq_B A'$. Since every algebraic strong extension of an element in $\mathcal{K}^{\mu}$ lies again in $\mathcal{K}^{\mu}$, it follows that rich structures are algebraically closed fields.

Corollary 9.3. *There is a (unique upto isomorphism) countable rich structure in $\mathcal{K}^{\mu}$. Moreover, all rich structures are $\mathcal{L}_{\infty,\omega}$-equivalent.*

10. Axioms for T^{μ}

Recall that given $\bar{a}, B \subseteq M \in \mathcal{K}$ we say that B is strong in M if $\langle B \rangle \leq M$ and we denote by $\delta(\bar{a}/B)$ the quantity $\delta(\langle B\bar{a} \rangle / \langle B \rangle)$. Let T^{μ} be the elementary theory of rich fields in $\mathcal{K}^{\mu}$. We will show in this section that T^{μ} is axiomatizable and model-complete.

Definition 10.1. Let $M \models T^{\mu}$ and B be some subset of M. Denote by $\mathrm{cl}_d^M(B)$ the collection of all f.g. $A \subseteq M$ with $\delta(A/\mathrm{cl}(B)) = 0$. Set $d_M(A/B) := d(A/B) := \delta(\mathrm{cl}(\langle A, B \rangle)/\mathrm{cl}(B))$.

Note that $\mathrm{cl}_d^M(B) = \{a \in M : d(a/B) = 0\}$.

It is easy to see [16] that:

Lemma 10.2. *For any structure in $\mathcal{K}$ the following holds:*

(1) $d(\bar{a}\bar{c}/B) = d(\bar{a}/B\bar{c}) + d(\bar{c}/B)$.
(2) *The closure operator* cl_d *defines a pregeometry over the set* $\ddot{\mathrm{U}}$ *of green points whose associated dimension function is* d.

Lemma 10.3. *Let $e \geq 0$, a subset $B = \langle B \rangle \leq M \in \mathcal{K}$ and a tuple $\bar{a} \in M$. Then:*

(1) *If $\delta(\bar{a}/B) = e$, then there is some existential $\mathcal{L}^*$-formula $\tau_\delta(\bar{x}, \bar{z})$ and a tuple $\bar{b} \in B$ such that:*
    - $\models \tau_\delta(\bar{a}, \bar{b})$,
    - *for every $\bar{a}'$ and $\bar{b}' \in B' \subseteq M' \in \mathcal{K}$ with $\models \tau_\delta(\bar{a}', \bar{b}')$, then $\delta(\bar{a}'/B') \leq e$.*
(2) *If $d(\bar{a}/B) = e$, then there is an existential $\mathcal{L}^*$-formula $\tau_d(\bar{x}, \bar{z})$ and a $\bar{b} \in B$ such that:*
    - $\models \tau_d(\bar{a}, \bar{b})$,
    - *for every $\bar{a}'$ and alle $\bar{b}' \in M' \in \mathcal{K}$ with $\models \tau_d(\bar{a}', \bar{b}')$, then $d(\bar{a}'/\bar{b}') \leq e$.*

Proof. We need only prove part (1). Hence, choose $\bar{a} \in M$ and $B \leq M$ as above. Let $B = A_0 \leq A_1 \leq \ldots \leq A_n = \langle B\bar{a} \rangle =: A$ be the decomposition of $B \leq A$ into minimal strong embeddings. Then $e = \delta(\bar{a}/B) = \sum_{i=1}^n \delta(A_i/A_{i-1})$. Therefore we may assume that $n = 1$, that is, $B \leq A$ is minimal strong. Four cases may occur by Lemma 6.4. Cases (1)–(3) are easy so we may hence consider case (4), that is, a minimal prealgebraic extension. Let $\bar{c}$ be a green basis for A/B and $\bar{b} \in B$ with $\mathrm{l.dim}_{\mathbb{Q}}(\bar{a}\bar{c}/B) = \mathrm{l.dim}_{\mathbb{Q}}(\bar{a}\bar{c}/\bar{b})$ and $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} B$. Let $\alpha \in \mathcal{C}$ be unique encoding a/B. Choose some quantifier-free $\mathcal{L}^*$-formula $\tilde{\tau}(\bar{x}, \bar{y}, \bar{z})$ with $\models \tilde{\tau}(\bar{a}, \bar{c}, \bar{b})$ such that:

- The tuples $\bar{a}$ and $\bar{c}$ are interdefinable over $\bar{b}$ (explicitly given).
- $\models \tilde{\tau}(\bar{x}, \bar{y}, \bar{z}) \rightarrow \bigwedge_i \ddot{\mathrm{U}}(y_i)$.
- $\bar{c}$ realizes some $\varphi_\alpha(\bar{y}, \bar{b}_1)$, where $\bar{b}_1$ lies in $\mathrm{acl}(\bar{b})$ (explicitly given).

By property (e) the formula $\tau_\delta(\bar{x}, \bar{z}) := \exists \bar{y}\, \tilde{\tau}(\bar{x}, \bar{y}, \bar{z})$ satisfies the required conditions, because $\delta(\bar{a}'/\bar{b}') = \delta(\bar{c}'/\bar{b}') \leq \delta(\bar{c}'/\mathrm{acl}(\bar{b}') \leq 0$ if $\models \tilde{\tau}(\bar{a}', \bar{c}', \bar{b}')$.

The general case may be reduced to the minimal one by considering some tuple to express the decomposition into minimal extensions. □

Given $M \subseteq N$ both elements in $\mathcal{K}$ such that M is $\mathcal{L}^*$-existentially closed in N, it follows from 10.3 that $M \leq N$: otherwise, there is a tuple $\bar{a} \in M$ with $d_M(\bar{a}) > d_N(\bar{a}) = e$. Choose some τ_d as in 10.3 such that $N \models \tau_d(\bar{a}, \bar{b})$ for some $\bar{b} \in \langle \emptyset \rangle \subseteq M$. Hence, $M \models \tau_d(\bar{a}, \bar{b})$ (since M is existentially closed in N), which contradicts $d_M(\bar{a}/\bar{b}) > e$. Hence, the following holds:

Lemma 10.4. *If M is an elementary submodel of $N \vDash T^\mu$, then M is strong in N.* □

Define now an $\mathcal{L}^*$-theory $\tilde{T}^\mu$ as follows:

(1) Every model lies in $\mathcal{K}^\mu$.
(2) Every model is an algebraically closed field of characteristic 0.
(3) Given a model M, a good code α and a parameter $\bar{b}$ in M, then there is no extension of M in $\mathcal{K}^\mu$ given by a green M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$.
(4) In some ω-saturated model there are infinitely many d-independent green generic elements.

Poizat [16] axiomatized universally condition "$\emptyset \leq M$": since $\psi_\alpha(\bar{x}_0, \ldots, \bar{x}_\nu)$ are quantifier-free, then axiom (1) is universal. Both ACF_0 and hence (2) are inductively axiomatizable. Corollary 8.4 yields the inductively axiomatization of (3), and the $\exists\forall$-axiomatization of (4) follows from Lemma 10.3.

A key result is the following:

Theorem 10.5. *An $\mathcal{L}^*$-structure M is rich if and only if it is an ω-saturated model of $\tilde{T}^\mu$. In particular, $\tilde{T}^\mu = T^\mu$ and $\tilde{T}^\mu$ is complete.*

Proof. The proof is divided into two parts: first, we show that every ω-saturated model of $\tilde{T}^\mu$ is rich in $\mathcal{K}^\mu$. Then show that all rich structures are models of $\tilde{T}^\mu$, which yields ω-saturation of rich structures because they are all ∞-equivalent by Corollary 9.3.

Hence, let M be an ω-saturated model of $\tilde{T}^\mu$, a finite subset $B \leq M$ and $A \geq B$ a f.g. structure in $\mathcal{K}^\mu$. We need to embedd A in M strong over B. We may assume that A/B is minimal strong. By Lemma 6.4 there are four possibilities:

A/B is algebraic. we are done by axiom (2).

A/B is minimal prealgebraic. Consider the free amalgam M' of M and A over B. Moreover, let α be the good code encoding A/B. Axiom (3) implies that M' is not in $\mathcal{K}^\mu$. Since $\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings by Theorem 9.2, then A must be already embeddable in M over B.

A/B is green generic, i.e. generated by a green transcendental element a. Axiom (4) and ω-saturation imply that M contains infinitely many green d-independent generic elements $(g_i)_{i\in\mathbb{N}}$. Since $d(B) = e < \infty$, there is some $i \in \mathbb{N}$ with $d(g_i/B) = 1$. Hence, $\langle Bg_i\rangle/B$ is a green generic extension (which lies in M), and the map $a \mapsto e_i$ yields a strong embedding of A in M over B.

A/B is white generic, i.e. generated by a white generic element w over B and $\ddot{\mathrm{U}}(A) = \ddot{\mathrm{U}}(B)$. It is easy to see that the sum w' of two B-generic d-independent green elements g_1 and g_2 is white B-generic, that is, $d(w'/B) = 2$. As above we can find such elements g_1, g_2 in M, so we are done.

Now, let M be a rich structure in $\mathcal{K}^\mu$. In order to show that $M \models T^\mu$, we first show that M is an algebraically closed field. Let $a \in \mathrm{acl}(M)$ and $B \leq M$ f.g. with $a \in \mathrm{acl}(B)$. If a is green, then a in B, because $B \leq M$. Otherwise, a is white and $\ddot{\mathrm{U}}(B) = \ddot{\mathrm{U}}(\langle Ba\rangle)$. Hence, the extension $\langle Ba\rangle \geq B$ is in $\mathcal{K}^\mu$, so we can realize it in M over B.

For axiom (3), consider a good code α and some parameter $\bar{b} \in M$. Let $\bar{a}$ be an M-generic realization of $\varphi_\alpha(\bar{x}, \bar{b})$. If $\langle M\bar{a}\rangle \in \mathcal{K}^\mu$, then choose some $B_0 \leq M$ containing $\bar{b}$. Therefore, $\langle B_0\bar{a}\rangle \in \mathcal{K}^\mu$ and by richness of M, we can find some $\bar{a}_0 \in M$ with $\langle B_0\bar{a}\rangle \cong \langle B_0\bar{a}_0\rangle =: B_1 \leq M$. Iterating this process with B_1, we can find a sequence $B_0 \leq B_1 \leq B_2 \leq \ldots$ in M, such that $B_{i+1} := \langle B_i\bar{a}_i\rangle \cong \langle B_i\bar{a}\rangle$. Then $\bar{a}_1, \bar{a}_2, \ldots$ is a sufficiently long green Morley sequence for $\varphi_\alpha(\bar{x}, \bar{b})$, whose difference realizes ψ_α, contradicting $M \in \mathcal{K}^\mu$.

Axiom (4) is satisfied by M by richness. □

Recall that given $A \subseteq M \in \mathcal{K}$, we define $\mathrm{cl}(A)$ as the smallest $\langle\cdot\rangle$-closed subset containing A with $\mathrm{cl}(A) \leq M$. If A is f.g., so is $\mathrm{cl}(A)$.

Corollary 10.6. *T^μ is complete. Two tuples $\bar{a}$ and $\bar{a}'$ in two models M and M' have the same type if and only if there is some $\mathcal{L}^*$-isomorphism f from $\mathrm{cl}_M(\bar{a})$ to $\mathrm{cl}_{M'}(\bar{a}')$, mapping $\bar{a}$ to $\bar{a}'$.*

Proof. The rich field obtained in Section 9 is a model of T^μ by Theorem 10.5. Given two models M and M' of T^μ, we may replace them by elementary extension and assume that they are ω-saturated. Hence they are rich by Theorem 10.5, hence elementarily equivalent by Corollary 9.3. So are M and M'.

In order to prove the second statement, suppose that both M and M' are ω-saturated, since by Lemma 10.4 the closure $\mathrm{cl}_M(\bar{a})$ does not change. Hence, M and M' are rich structures and the map $f : \mathrm{cl}_M(\bar{a}) \to \mathrm{cl}_{M'}(\bar{a}')$ induces a back-and-forth system, so f is elementary.

Suppose now that $\bar{a}$ in M has the same type as $\bar{a}'$ in M'. Since $\mathrm{cl}(\bar{a})$ is in the algebraic closure (in T^μ) of $\bar{a}$, there is an elementary map f from $\mathrm{cl}(\bar{a})$ to M' by ω-saturation, mapping $\bar{a}$ onto $\bar{a}'$. Let $A' = f(\mathrm{cl}(\bar{a}))$. Hence $A' \leq M'$ because A' has the same type as $\mathrm{cl}(\bar{a}')$. Therefore, $A' = \mathrm{cl}(\bar{a}')$. □

Corollary 10.7. *The theory T^μ is model-complete.*

Proof. We will give a straight-foward proof due to M. Ziegler. We need only show that given any two models M and N of T^μ with $M \subseteq N$ then M is strong in N. This implies that $\mathrm{cl}_M(\bar{a}) = \mathrm{cl}_N(\bar{a})$ for any $\bar{a}$ in M. So the inclusion is elementary by Corollary 10.6. In particular, we will show the following:

Claim. *If $M \models T^\mu$ and $M \subseteq N \in \mathcal{K}^\mu$, then M is strong in N.*

Otherwise, choose some $M \subseteq N_1$ with minimal $\mathrm{l.dim}_{\mathbb{Q}}(N_1/M) = e$. Since $M = \mathrm{acl}(M)$, then $e \geq 2$. Minimality of e implies that $\delta(N_0/M) \geq 0$ for every $N_0 = \langle N_0 \rangle$ with $M \subseteq N_0 \subsetneq N_1$. In particular, $M \leq N_0$. Choose now some N_0 with $\mathrm{l.dim}_{\mathbb{Q}}(N_0/M) = e - 1$.

$$-1 \geq \delta(N_1/M) = \delta(N_1/N_0) + \delta(N_0/M)$$

and $\delta(N_1/N_0) \geq -1$ imply that $\delta(N_0/M) \leq 0$. Since $M \leq N_0$, the extension N_0/M is prealgebraic in $\mathcal{K}^\mu$ (i.e. a tower of algebraic and minimal prealgebraic extensions). Hence, find N'/M minimal prealgebraic with $M \leq N' \leq N_0$, which contradicts axiom (3). □

Note 10.8. One can show that axiom (4) follows already from (1)–(3), by aproximating a green generic extension by suitable prealgebraic extensions. On the other hand, the $\forall\exists$-axiomatization follows from model-completeness (Corollary 10.7) by general model-theoretical arguments.

11. Ranks

We show in this section that T^μ has Morley rank 2. Let acl^μ denote the algebraic closure in models of T^μ. All model-theoretical notions refer exclusively to T^μ, which we will emphazise with μ if necessary. We will show that $\mathrm{acl}^\mu(\bar{a})$ is the union of all minimal prealgebraic extensions of $\mathrm{cl}(\bar{a})$.

Lemma 11.1. *Both closures acl^μ and cl_d^M agree in models of T^μ.*

Proof. If B is f.g., so is $\mathrm{cl}(B)$ and contained in $\mathrm{acl}^\mu(B)$. Hence, we may assume that B is f.g. and strong in $M \models T^\mu$.

First, show that $\mathrm{cl}_d^M(B) \subseteq \mathrm{acl}^\mu(B)$. Let $A \subset M$ be f.g. with $\delta(A/B) = 0$. Since $\mathrm{l.dim}_{\mathbb{Q}}(A/B)$ is finite, then we may decompose A/B into a finite sequence of minimal extensions. If $A' \supseteq B$ is such that $\delta(A'/B) = 0$, then $A' \leq M$ because

$B \leq M$. Hence, we may assume that A/B is minimal. By Lemma 6.4 one of the following holds:

i) A is algebraic over B (in the field reduct). Hence, $A \subseteq \mathrm{acl}^\mu(B)$.
ii) A is minimal prealgebraic over B. Choose a good code α and some parameter $\bar{b}$ in $\mathrm{acl}(B)$ encoding A/B by Theorem 4.10. Then, $A = \langle B\bar{a}\rangle$ for some generic green realization $\bar{a} \models \varphi_\alpha(\bar{x}, \bar{b})$. We need only show that all green realizations of $\varphi_\alpha(\bar{x}, \bar{b})$ lie already in M, for otherwise there is some $M \preccurlyeq N$ and $\bar{a}' \in N$ not completely contained in M. Hence, $\bar{a}'$ is generic over M cotradicting axiom (3).

For the other inclusion, choose some $a \in M \setminus \mathrm{cl}_d^M(B)$. Set $A = \mathrm{cl}(B, a)$ and observe that $\delta(A/B) > 0$. Decompose now A/B in minimal extensions $B \leq A_0 \leq A_1 \leq \ldots \leq A_n = A$. Then there is some $i < n$ with $\delta(A_{i+1}/A_i) > 0$. By Lemma 6.4 we obtain $\mathrm{l.dim}_{\mathbb{Q}}(A_{i+1}/A_i) = 1$ so the extension A_{i+1}/A_i is either white or green generic. Corollary 8.3 implies that the free amalgam of A_{i+1} and every strong extension of A_i lies in $\mathcal{K}^\mu$. Richness of M implies that there are infinitely many $A' \leq M$, isomorphic to A_{i+1} over A_i. Moreover, they all have the same type over A_i by Corollary 10.6. So, $A_{i+1} \not\subseteq \mathrm{acl}^\mu(A_i)$, hence $a \notin \mathrm{acl}^\mu(B)$, because A_{i+1} is algebraic over $\langle B, a\rangle$ and $B \subseteq A_i$. □

Theorem 11.2. *T^μ has Morley rank 2 and is uncountably categorical. The white generic type has Morley rank 2 and the green generic one has Morley rank 1. Algebraic closure is determined by cl_d-closure in any model of T^μ. Moreover, for all $\bar{a}$ and B we have that :*

$$\mathrm{MR}(\bar{a}/B) = \mathrm{U}(\bar{a}/B) = d(\bar{a}/B).$$

Proof. Let M be an ω-saturated model of T^μ, seen as subset of the monster model of T^μ. We compute $\mathrm{MR}(a/M)$ for elements a coming from the monster model. Since

$$0 \leq d(a/M) \leq \delta(a/M) \leq 2.$$

there are four cases to consider:

$d(a/M) = 0$: By Lemma 10.4, we have that $a \in \mathrm{acl}^\mu(M) = M$. So $\mathrm{MR}(a/M) = 0$.

$d(a/M) = 1$ and a is green: Then $\langle Ma\rangle$ is strong in the monster model and $\mathrm{tp}(a/M)$ is the type of the *green generic* element by Corollary 10.6. Since all other green generic types are algebraic, then $\mathrm{MR}(a/M) = 1$, so $\ddot{\mathrm{U}}$ defines a strongly minimal set.

$d(a/M) = 1$ with a white: There must be some green point $c \in \mathrm{cl}(\langle M, a\rangle) \setminus M$. Hence, $\langle Mc\rangle \leq \mathrm{cl}(Ma)$ and $d(a/Mc) = 0$. Therefore, a and c are T^μ-interalgebraic over M and by the above case, $\mathrm{MR}(a/M) = \mathrm{MR}(c/M) = 1$.

$d(a/M) = 2$: Then $\ddot{\mathrm{U}}(\langle Ma\rangle) = \ddot{\mathrm{U}}(M)$ and $\langle Ma\rangle$ is strong in the monster model. Corollary 10.6 implies that $\mathrm{tp}(a/M)$ is the type of the *white generic element*, that is, the *generic type* of the field. Since all other types have Morley rank at most 1, then $\mathrm{MR}(a/M) \leq 2$. Now, $\ddot{\mathrm{U}}(M)$ is an infinite group of infinite index in M. Therefore, $\mathrm{MR}(a/M) = 2$ and T^μ has Morley rank 2.

For the last statement, $\mathrm{MR}(\bar{a}/B) = \mathrm{U}(\bar{a}B) = d(\bar{a}/B)$, recall that $\mathrm{MR} = \mathrm{U}$ holds on all $\aleph_1$-categorical theories. So Morley rank is additive and the above shows that $\mathrm{MR}(a/B) = d(a/B)$ for elements. Since d is also additive, then we are done. □

Note 11.3. Following an idea of Poizat [16], it is possible to construct, for every natural number $r \geq 2$, a field of Morley rank r with a strongly minimal multiplicative green subgroup, working with the following predimension:

$$\delta(A) = r\dim(A) - (r-1)\,\mathrm{l.dim}_{\mathbb{Q}}(\ddot{\mathrm{U}}(A)).$$

Moreover, as in [6], the predimension

$$\delta(A) = r\dim(A) - \mathrm{l.dim}_{\mathbb{Q}}(\ddot{\mathrm{U}}(A))$$

leeds to a field of Morley rank r with a multiplicative green subgroup of Morley rank $r-1$.

Question 11.4. J. Kirby [12] generalised Theorem 2.2 to semiabelian varieties. Can this be used to construct a field of finite Morley rank with a non-algebraic subgroup of an arbitrary semiabelian variety?

References

[1] J. Ax, *On Schanuel's Conjecture*, Annals of Mathematics **93** (1971), 252–258.
[2] J. Baldwin, K. Holland, *Constructing ω-stable structures: fields of rank 2*, J. Symb. Logic, **65**, n^o 1, (2000), 371–391.
[3] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *Hruhovski's Fusion*, to appear at the special volume on Felgner's 65$^{\text{th}}$ birthday on Studies in Logic, King's College Publications, London, U.K., (2007).
[4] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *Fusion over a vector space*, J. Math. Logic., **6**, vol 2, (2007) 141–162. (2005).
[5] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *Red fields*, J. Symb. Logic, **72**, vol 1, (2007), 207–225.
[6] A. Baudisch, A. Martin-Pizarro, M. Ziegler, *On fields and colours*, Algebra i Logika, **45**, n^o 2, (2006), 92–105.
[7] L. van den Dries, K. Schmidt, *Bounds in the theory of polynomial rings over fields*, Invent. Math. **76** (1984), 77–91.
[8] J. B. Goode, *Hrushovski's geometries*, Seminarberichte der Humboldt Univ. n^o 104, (1989), 106–117.
[9] A. Hasson, M. Hils, *Fusion over sublanguages*, J. Symb. Logic **71** (2006), n^o 2, 361–398.
[10] E. Hrushovski, *Strongly minimal expansions of algebraically closed fields*, Israel J. Math, **79** (1992), 129–151.
[11] E. Hrushovski, *A new strongly minimal set*, Annals of Pure and Applied Logic **62** (1993), 147–166.
[12] J. Kirby, *The theory of the exponential differential equations of semiabelian varieties*, arXiv:math.AG/0708.1352.
[13] A. Macintyre, *On ω_1-categorical fields*, Fund. Math.,**71** (1971), 1-25.
[14] E. Mustafin, *Structure des groupes linéaires définissables dans un corps de rang de Morley fini*, J. Algebra, **281**, (2004), n^o 2, 753–773.
[15] B. Poizat, *Le carré de l'égalité* , J. Symb. Logic, **64**, n^o 3 (1999), 1338–1355.
[16] B. Poizat, *L'égalité au cube*, J. Symb. Logic, **66**, n^o 4 (2001), 1647–1676.
[17] F. O. Wagner, *Fields of finite Morley Rank*, J. Symb. Logic, **66** n^o 2 (2001), 703–706.
[18] F. O. Wagner, *Bad fields in positive characteristic*, Bull. London Math. Soc., **35** (2003), 499–502.
[19] M. Ziegler, *A note on generic types*, arXiv:math.LO/0608433.
[20] B. I. Zilber, *Groups and rings whose theory is categorical*, Fund. Math. **95** (1977), 173–188.

Institut für Mathematik, Humboldt-Universität zu Berlin, D-10099 Berlin, Germany.
baudisch@mathematik.hu-berlin.de

Institut für Mathematik, Humboldt-Universität zu Berlin, D-10099 Berlin, Germany.
Equipe de Logique Mathématique, Université Denis-Diderot Paris VII, 75251 Paris Cedex 05, France.
Université de Lyon; Université Lyon 1; CNRS, Institut Camille Jordan, 43 bd du 11 nov. 1918, 69622 Villeurbanne Cedex, France.
hils@mathematik.hu-berlin.de

Université de Lyon; Université Lyon 1; CNRS, Institut Camille Jordan, 43 bd du 11 nov. 1918, 69622 Villeurbanne Cedex, France.
Institut für Mathematik, Humboldt-Universität zu Berlin, D-10099 Berlin, Germany.
pizarro@math.univ-lyon1.fr

Université de Lyon; Université Lyon 1; CNRS, Institut Camille Jordan, 43 bd du 11 nov. 1918, 69622 Villeurbanne Cedex, France.
wagner@math.univ-lyon1.fr