Tameness in non-archimedean geometry through model theory (after Hrushovski-Loeser)

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Model Theory 2013
Ravello (Italy), 10th – 15th June 2013

Outline

Valued fields: basics and notation

Let K be a field and $\Gamma = (\Gamma, 0, +, <)$ an ordered abelian group.

A map $\mathrm{val} : K \to \Gamma_\infty = \Gamma \dot{\cup} \{\infty\}$ is a **valuation** if it satisfies

1. $\mathrm{val}(x) = \infty$ iff $x = 0$;
2. $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$;
3. $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.

(Here, ∞ is a distinguished element $> \Gamma$ and absorbing for $+$.)

- $\Gamma = \Gamma_K$ is called the **value group**.
- $\mathcal{O} = \mathcal{O}_K = \{x \in K \mid \mathrm{val}(x) \geq 0\}$ is the **valuation ring**, with (unique) maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \{x \mid \mathrm{val}(x) > 0\}$;
- $\mathrm{res} : \mathcal{O} \to k = k_K := \mathcal{O}/\mathfrak{m}$ is the **residue map**, and k_K is called the **residue field**.

The valuation topology

Let K be a valued field with value group Γ.

- For $a \in K$ and $\gamma \in \Gamma$ let $B_{\geq\gamma}(a) := \{x \in K \mid \mathrm{val}(x - a) \geq \gamma\}$ be the **closed ball** of (valuative) radius γ around a.
- Similarly, one defines the **open ball** $B_{>\gamma}(a)$.
- The open balls form a basis for a topology on K, called the **valuation topology**, turning K into a topological field.
- Both the 'open' and the 'closed' balls are clopen sets in the valuation topology. In particular, K is totally disconnected.
- Let V be an algebraic variety defined over K.
  Using the product topology on K^n and gluing, one defines the valuation topology on $V(K)$ (also totally disconnected).

Fields with a (complete) non-archimedean absolute value

Assume that K is a valued field such that $\Gamma_K \leq \mathbb{R}$.

- $|\cdot|: K \to \mathbb{R}_{\geq 0}$, $|x| := e^{-\mathrm{val}(x)}$, defines an absolute value.
- $(K, |\cdot|)$ is non-archimedean, and any field with a non-archimedean absolute value is obtained in this way.
- $(K, |\cdot|)$ is called **complete** if it is complete as a metric space, i.e. if every Cauchy sequence has a limit in K.

Examples of complete non-archimedean fields

- $\mathbb{Q}_p$ (the field p-adic numbers), and any finite extension of it
- $\mathbb{C}_p = \widehat{\mathbb{Q}_p^a}$ (the p-adic analogue of the complex numbers)
- $k((t))$, with the t-adic absolute value (k any field)
- k with the trivial absolute value ($|x| = 1$ for all $x \in k^\times$)

Non-archimedean analytic geometry

- For K a complete non-archimedean field, one would like to do analytic geometry over K similarly to the way one does analytic geometry over $\mathbb{C}$, with a 'nice' underlying topological space.
- There exist various approaches to this, due to Tate (rigid analytic geometry), Raynaud, Berkovich, Huber etc.

Berkovich's approach: **Berkovich (analytic) spaces** (late 80's)

- provide spaces endowed with an actual topology (not just a Grothendieck topology), in which one may consider paths, singular (co-)homology etc.;
- are obtained by **adding points** to the set of naive points of an analytic / algebraic variety over K;
- have been used with great success in many different areas.

Berkovich spaces in a glance

We briefly describe the Berkovich analytification (as a topological space) V^{an} of an affine algebraic variety V over K.

- Let $K[V]$ be the ring of regular functions on V. As a set, V^{an} equals the set of **multiplicative seminorms** $|\cdot|$ on $K[V]$ ($|fg| = |f| \cdot |g|$ and $|f+g| \leq \max(|f|, |g|)$) which extend $|\cdot|_K$.
- $V(K)$ may be identified with a subset of V^{an}, via $a \mapsto |\cdot|_a$, where $|f|_a := |f(a)|_K$.
- Note $V^{an} \subseteq \mathbb{R}^{K[V]}$. The topology on V^{an} is defined as the induced one from the product topoloy on $\mathbb{R}^{K[V]}$.

Remark

Let $(L, |\cdot|_L)$ be a normed field extension of K, and let $b \in V(L)$. Then b corresponds to a map $\varphi : K[V] \to L$, and $|\cdot|_b \in V^{an}$, where $|f|_b = |\varphi(f)|_L$. Moreover, any element of V^{an} is of this form.

A glimpse on the Berkovich affine line

Example

Let $V = \mathbb{A}^1$, so $K[V] = K[X]$.

- For any $r \in \mathbb{R}_{\geq 0}$, we have $\nu_{0,r} \in \mathbb{A}^{1,an}$, where

$$|\sum_{i=0}^{n} c_i X^i|_{\nu_{0,r}} := \max_{0 \leq i \leq n} \left(|c_i|_K \cdot r^i \right).$$

- $\nu_{0,0}$ corresponds to $0 \in \mathbb{A}^1(K)$, and $\nu_{0,1}$ to the *Gauss norm*.
- The map $r \mapsto \nu_{0,r}$ is a continuous path in $\mathbb{A}^{1,an}$.
- In fact, the construction generalises suitably, showing that $\mathbb{A}^{1,an}$ is contractible.

Topological tameness in Berkovich spaces

Berkovich spaces have excellent general topological properties, e.g. they are **locally compact** and **locally path-connected**.

Using deep results from algebraic geometry, various **topological tameness** properties had been established, e.g.:

- Any compact Berkovich space is **homotopic to a (finite) simplicial complex** (Berkovich);
- Smooth Berkovich spaces are **locally contractible** (Berkovich).
- If V is an algebraic variety, 'semi-algebraic' subsets of V^{an} have **finitely many connected components** (Ducros).

Hrushovski-Loeser's work: main contributions

Foundational

- They develop **'non-archimedean (rigid) algebraic geometry'**, constructing a 'nice' space $\widehat{V}$ for an algebraic variety V over any valued field K,
  - with no restrictions on the value group Γ_K;
  - no need to work with a complete field K.
- **Entirely new methods**: the geometric model theory of ACVF is shown to be perfectly suited to address topological tameness (combining stability and o-minimality).

Applications to Berkovich analytifications of algebraic varieties

They obtain **strong topological tameness results** for V^{an},

- without smoothness assumption on the variety V, and
- avoiding heavy tools from algebraic geometry.

Valued fields as first order structures

- There are various choices of languages for valued fields.
- $\mathcal{L}_{\mathrm{div}} := \mathcal{L}_{rings} \cup \{\mathrm{div}\}$ is a language with only one sort **VF** for the valued field.
- A valued field K gives rise to an $\mathcal{L}_{\mathrm{div}}$-structure, via

$$x \,\mathrm{div}\, y :\Leftrightarrow \mathrm{val}(x) \leq \mathrm{val}(y).$$

- $\mathcal{O}_K = \{x \in K : 1 \,\mathrm{div}\, x\}$, so $\mathcal{O}_K$ is $\mathcal{L}_{\mathrm{div}}$-definable $\Rightarrow$ the valuation is encoded in the $\mathcal{L}_{\mathrm{div}}$-structure.
- **ACVF**: theory of **alg. closed non-trivially valued fields**

QE in algebraically closed valued fields

Fact (Robinson)

The theory ACVF *has QE in* $\mathcal{L}_{\mathrm{div}}$*. Its completions are given by* $\mathrm{ACVF}_{p,q}$*, for* $(p,q) = (char(K), char(k))$*.*

Corollary

1. *In ACVF, a set is definable iff it is* **semi-algebraic***, i.e. a finite boolean combination of sets given by conditions of the form* $f(\overline{x}) = 0$ *or* $\mathrm{val}(f(\overline{x})) \leq \mathrm{val}(g(\overline{x}))$*, where* f*,* g *are polynomials.*
2. *Definable sets in 1 variable are (finite) boolean combinations of singletons and balls.*
3. ACVF *is* **NIP***, i.e., there is no formula* $\varphi(\overline{x}, \overline{y})$ *and tuples* $(\overline{a}_i)_{i\in\mathbb{N}}$*,* $(\overline{b}_J)_{J\subseteq\mathbb{N}}$ *(in some model) such that* $\varphi(\overline{a}_i, \overline{b}_J)$ *iff* $i \in J$*.*

A variant: valued fields in a three-sorted language

Let $\mathcal{L}_{k,\Gamma}$ be the following 3-sorted language, with sorts **VF** for the valued field, Γ_∞ and **k**:

- Put $\mathcal{L}_{rings}$ on $K = \mathbf{VF}$, $\{0, +, <, \infty\}$ on Γ_∞ and $\mathcal{L}_{rings}$ on $\mathbf{k}$;
- $\mathsf{val} : K \to \Gamma_\infty$, and
- $\mathrm{RES} : K \to \mathbf{k}$ as additional function symbols.

A valued field K is naturally an $\mathcal{L}_{k,\Gamma}$-structure, via

$$\mathrm{RES}(x, y) := \begin{cases} \mathrm{res}(xy^{-1}), & \text{if } \mathsf{val}(x) \geq \mathsf{val}(y) \neq \infty; \\ 0 \in k, & \text{else.} \end{cases}$$

ACVF in the three-sorted language

Fact

ACVF eliminates quantifiers in $\mathcal{L}_{k,\Gamma}$.

Corollary

In ACVF, the following holds:

1. Γ *is a* **pure divisible ordered abelian group**: *any definable subset of* Γ^n *is* $\{0,+,<\}$*-definable (with parameters from* Γ*). In particular,* Γ *is o-minimal.*
2. $\mathbf{k}$ *is a* **pure ACF**: *any definable subset of* $\mathbf{k}^n$ *is* $\mathcal{L}_{rings}$*-definable.*
3. $\mathbf{k} \perp \Gamma$, *i.e. every definable subset of* $\mathbf{k}^m \times \Gamma^n$ *is a finite union of rectances* $D \times E$.
4. *Any definable function* $f : K^n \to \Gamma_\infty$ *is piecewise of the form* $f(\overline{x}) = \frac{1}{m}[\mathrm{val}(F(\overline{x})) - \mathrm{val}(G(\overline{x}))]$, *for* $F, G \in K[\overline{x}]$ *and* $m \geq 1$.

A description of 1-types over models of ACVF

Let $K \preccurlyeq \mathbb{U} \models \mathrm{ACVF}$, with $\mathbb{U}$ suff. saturated. A K-(type-)definable subset $B \subseteq \mathbb{U}$ is a **generalised ball over** K if B is equal to one of the following:

- a singleton $\{a\} \subseteq K$;
- a closed ball $B_{\geq\gamma}(a)$ ($a \in K$, $\gamma \in \Gamma_K$);
- an open ball $B_{>\gamma}(a)$ ($a \in K$, $\gamma \in \Gamma_K$);
- a (non-empty) intersection $\bigcap_{i\in I} B_i$ of K-definable balls B_i with no minimal B_i;
- $\mathbb{U}$.

Fact

By QE, we have $S_1(K) \overset{1:1}{\leftrightarrow} \{$*generalised balls over* $K\}$*, given by*

- $p = \mathrm{tp}(t/K) \mapsto Loc(t/K) := \bigcap b$*, where b runs over all generalised balls over* K *containing* t*;*
- $B \mapsto p_B \mid K$*, where* $p_B \mid K$ *is the* **generic type in** B *expressing* $x \in B$ *and* $x \notin b'$ *for any* K*-def. ball* $b' \subsetneq B$*.*

Context

- $\mathcal{L}$ is some language (possibly many-sorted);
- T is a **complete** $\mathcal{L}$-theory with QE;
- $\mathbb{U} \models T$ is a fixed **universe** (i.e. very saturated and homogeneous);
- all models M (and all parameter sets A) we consider are **small**, with $M \preccurlyeq \mathbb{U}$ (and $A \subseteq \mathbb{U}$).

Imaginary Sorts and Elements

- Let E is a definable equivalence relation on some $D \subseteq_{def} \mathbb{U}^n$. If $d \in D(\mathbb{U})$, then d/E is an **imaginary** in $\mathbb{U}$.
- If $D = \mathbb{U}^n$ for some n and E is $\emptyset$-definable, then U^n/E is called an **imaginary sort**.
- Recall: **Shelah's *eq*-construction** is a canonical way to pass from $\mathcal{L}, M, T$ to $\mathcal{L}^{eq}, M^{eq}, T^{eq}$, adding a new sort (and a quotient function) for each imaginary sort.
- Given $\varphi(x,y)$, let $E_\varphi(y,y') := \forall x[\varphi(x,y) \leftrightarrow \varphi(x,y')]$. Then b/E_φ may serve as a **code** $\ulcorner W \urcorner$ for $W = \varphi(\mathbb{U}, b)$.

Example

Consider $K \models \mathrm{ACVF}$ (in $\mathcal{L}_{\mathrm{div}}$).

- $\mathbf{k}, \Gamma \subseteq K^{eq}$, i.e. $\mathbf{k}$ and Γ are imaginary sorts.
- More generally, $\mathcal{B}^o, \mathcal{B}^{cl} \subseteq K^{eq}$ (the set of open / closed balls).

Elimination of imaginaries

Definition (Poizat)

The theory T eliminates imaginaries if every imaginary element $a \in \mathbb{U}^{eq}$ is interdefinable with a real tuple $\overline{b} \in \mathbb{U}^n$.

Examples of theories which eliminate imaginaries

1. T^{eq} (for an arbitrary theory T)
2. ACF (Poizat)
3. The theory DOAG of non-trivial divisible ordered abelian groups (more generally every o-minimal expansion of DOAG)

Fact

ACVF *does not eliminate imaginaries in the 3-sorted language* $\mathcal{L}_{k,\Gamma}$ *(Holly), even if sorts for open and closed balls* $\mathcal{B}^{o}$ *and* $\mathcal{B}^{cl}$ *are added (Haskell-Hrushovski-Macpherson).*

The geometric sorts

- $s \subseteq K^n$ is a **lattice** if it is a free $\mathcal{O}$-submodule of rank n;
- for $s \subseteq K^n$ a lattice, $s/\mathfrak{m}s$ is a definable n-dimensional $\mathbf{k}$-vector space.

For $n \geq 1$, let

$$S_n := \{\text{lattices in } K^n\},$$

$$T_n := \dot{\bigcup}_{s \in S_n} s/\mathfrak{m}s.$$

Fact

1. *S_n and T_n are imaginary sorts, $S_1 \cong \Gamma$ (via $a\mathcal{O} \mapsto \mathrm{val}(a)$), and also $\mathbf{k} = \mathcal{O}/\mathfrak{m} \subseteq T_1$.*
2. *$S_n \cong \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$; for T_n, there is a similar description as a finite union of coset spaces.*

Classification of Imaginaries in ACVF

$\mathcal{G} = \{\mathbf{VF}\} \cup \{S_n,\ n \geq 1\} \cup \{T_n,\ n \geq 1\}$ are the geometric sorts.
Let $\mathcal{L}_{\mathcal{G}}$ be the (natural) language of valued fields in $\mathcal{G}$.

Theorem (Haskell-Hrushovski-Macpherson 2006)

ACVF eliminates imaginaries down to **geometric sorts**, *i.e. the theory ACVF considered in $\mathcal{L}_{\mathcal{G}}$ has EI.*

Convention

From now on, by ACVF we mean any completion of this theory, considered in the geometric sorts.

Moreover, any theory T we consider will be assumed to have EI.

The notion of a definable type

Definition

- Let $M \models T$ and $A \subseteq M$. A type $p(\overline{x}) \in S_n(M)$ is called A-**definable** if for every $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ there is an $\mathcal{L}_A$-formula $d_p\varphi(\overline{y})$ such that

$$\varphi(\overline{x}, \overline{b}) \in p \;\Leftrightarrow\; M \models d_p\varphi(\overline{b}) \quad (\text{for every } \overline{b} \in M)$$

- We say p is **definable** if it is definable over some $A \subseteq M$.
- The collection $(d_p\varphi)_\varphi$ is called a **defining scheme** for p.

Remark

If $p \in S_n(M)$ is definable via $(d_p\varphi)_\varphi$, then the same scheme gives rise to a (unique) type over any $N \succcurlyeq M$, denoted by $p \mid N$.

Definable types: first properties

- **(Realised types are definable)**
  Let $\overline{a} \in M^n$. Then $\mathrm{tp}(\overline{a}/M)$ is definable.
  (Take $d_p\varphi(\overline{y}) = \varphi(\overline{a}, \overline{y})$.)

- **(Preservation under algebraic closure)**
  If $\mathrm{tp}(\overline{a}/M)$ is definable and $\overline{b} \in \mathrm{acl}(M \cup \{\overline{a}\})$, then $\mathrm{tp}(\overline{b}/M)$ is definable, too.

- **(Transitivity)** Let $\overline{a} \in N$ for some $N \succcurlyeq M$, $A \subseteq M$. Assume
  - $\mathrm{tp}(\overline{a}/M)$ is A-definable;
  - $\mathrm{tp}(\overline{b}/N)$ is $A \cup \{\overline{a}\}$-definable.

  Then $\mathrm{tp}(\overline{a}\overline{b}/M)$ is A-definable.

  We note that the converse of this is false in general.

Definable 1-types in *o*-minimal theories

Let T be *o*-minimal (e.g. $T = \mathrm{DOAG}$) and $M \models T$.

- Let $p(x) \in S_1(M)$ be a non-realised type.
- Recall that p is determined by the cut $C_p := \{d \in M \mid d < x \in p\}$.
- Thus, by *o*-minimality, $p(x)$ is definable
  $\Leftrightarrow d_p\varphi(y)$ exists for $\varphi(x, y) := x > y$
  $\Leftrightarrow C_p$ is a definable subset of M
  $\Leftrightarrow C_p$ is a rational cut
- e.g. in case $C_p = M$, $d_p\varphi(y)$ is given by $y = y$;
- in case $C_p =]-\infty, \delta]$, $d_p\varphi(y)$ is given by $y \leq \delta$
  ($p(x)$ expresses: x is "just right" of δ; this p is denoted by δ^+).

Definable 1-types in ACVF

Fact

Let $K \models \mathrm{ACVF}$ and $p = \mathrm{tp}(t/K) \in S_1(K)$. TFAE:

1. $\mathrm{tp}(t/K)$ *is definable;*
2. *$Loc(t/K)$ is definable (and not just type-definable).*

Proof.

If $\mathrm{tp}(t/K)$ is definable, then the set of K-definable balls containing t is definable over K, so is its intersection. (2)⇒(1) is clear. □

For $t \notin K$, letting $L = K(t)$, we get three cases:

- L/K is a residual extension, i.e. $k_L \supsetneq k_K$. Then t is generic in a closed ball, so p is definable.

  [Indeed, replacing t by $at + b$, WMA $\mathrm{val}(t) = 0$ and $\mathrm{res}(t) \notin k_K$, so t is generic in $\mathcal{O}$.]

Definable 1-types in ACVF (continued)

- L/K is a ramified extension, i.e. $\Gamma_L \supsetneq \Gamma_K$. Up to a translation WMA $\gamma = \mathrm{val}(t) \notin \Gamma(K)$.

  p is definable $\Leftrightarrow$ the cut def. by $\mathrm{val}(t)$ in Γ_K is rational.

  [Indeed, p is determined by $p_\Gamma := \mathrm{tp}_{\mathrm{DOAG}}(\gamma/\Gamma_K)$, so p is definable $\Leftrightarrow$ p_Γ is definable.]

- L/K is an immediate extension, i.e. $k_K = k_L$ and $\Gamma_K = \Gamma_L$. Then p is not definable.

  [Indeed, in this case, letting $B := Loc(t/K)$, we get $B(K) = \emptyset$. In particular, B is not definable.]

Definability of types in ACF

Proposition

In ACF, all types over all models are definable.

Proof.

Let $K \models ACF$ and $p \in S_n(K)$.

Let $I(p) := \{f(\overline{x}) \in K[\overline{x}] \mid f(\overline{x}) = 0 \text{ is in } p\} = (f_1, \ldots, f_r)$.

By QE, every formula is equivant to a boolean combination of polynomial equations. Thus, it is enough to show:

For any d the set of (coefficients of) polynomials $g(\overline{x}) \in K[\overline{x}]$ of degree $\leq d$ such that $g \in I_p$ is definable. This is classical. □

Remark

The above result is a consequence of the **stability** *of ACF.*
In fact, it characterises stability.

Products of definable types

- Assume $p = p(x)$ and $q = q(y)$ are A-definable types.
- There is a unique A-definable type $p \otimes q$ in variables (x, y), constructed as follows: Let $b \models q \mid A$ and $a \models p \mid Ab$. Then

$$p \otimes q \mid A = \mathrm{tp}(a, b/A).$$

- The n-fold product $p \otimes \cdots \otimes p$ is denoted by $p^{(n)}$.

Remark

1. *$\otimes$ is associative.*
2. *$\otimes$ is in general not commutative, as is shown by the following: Let $p(x)$ and $q(y)$ both be equal to 0^+ in DOAG. Then $p(x) \otimes q(y) \vdash x < y$, whereas $q(y) \otimes p(x) \vdash y < x$.*
3. *In a stable theory, $\otimes$ corresponds to the non-forking extension, so $\otimes$ is in particular commutative.*

The stable part

Let T be given and $A \subseteq \mathbb{U}$ a parameter set.

Recall that an A-definable set D is **stably embedded** if every definable subset of D^n is definable with parameters from $D(\mathbb{U}) \cup A$.

Definition

- The **stable part over** A, denoted St_A, is the multi-sorted structure with a sort for each A-definable stable stably embedded set D and with the full induced structure (from $\mathcal{L}_A$).
- For $\overline{a} \in \mathbb{U}$, set $St_A(\overline{a}) := \mathrm{dcl}(A\overline{a}) \cap St_A$.

Fact

St_A is a stable structure.

The stable part in ACVF

Consider ACVF in $\mathcal{L}_{\mathcal{G}}$. Given A, we denote by $\mathrm{VS}_{\mathbf{k},A}$ the many sorted structure with sorts $s/\mathfrak{m}s$, where $s \in S_n(A)$ for some n.

Fact (HHM)

Let D be an A-definable set. TFAE:

1. *D is stable and stably embedded.*
2. *D is* **k-internal**, *i.e. there is a finite set $F \subseteq \mathbb{U}$ such that $D \subseteq \mathrm{dcl}(\mathbf{k} \cup F)$*
3. *$D \subseteq \mathrm{dcl}(A \cup \mathrm{VS}_{\mathbf{k},A})$*
4. *$D \perp \Gamma$ (def. subsets of $D^m \times \Gamma^n$ are finite unions of rectangles)*

Corollary

Up to interdefinability, St_A is equal to $\mathrm{VS}_{\mathbf{k},A}$. In particular, if $A = K \models \mathrm{ACVF}$, then St_A may be identified with $\mathbf{k}$.

Stable domination (in ACVF)

- Idea: a stably dominated type is 'generically' controlled by its stable part.
- To ease the presentation and avoid technical issues around base change, we will restrict the context and work in ACVF.

Definition

Let p be an A-definable type. We say p is **stably dominated** if for $\overline{a} \models p \mid A$ and every $B \supseteq A$ such that

$$\mathrm{St}_A(\overline{a}) \underset{A}{\downarrow} \mathrm{St}_A(B) \text{ (in the stable structure } \mathrm{St}_A = \mathrm{VS}_{\mathbf{k},A}),$$

we have $\mathrm{tp}(\overline{a}/A) \cup \mathrm{tp}(\mathrm{St}_A(\overline{a})/\mathrm{St}_A(B)) \vdash \mathrm{tp}(\overline{a}/B)$.

(We will then also say that $p \mid A = \mathrm{tp}(\overline{a}/A)$ is stably dominated.)

Fact

The above does not depend on the choice of the set A over which p is defined, so the notion is well-defined.

Stably dominated types inherit many nice properties from stable theories. Here is one:

Fact

If p is stably dominated type and q an arbitrary definable type, then $p \otimes q = q \otimes p$. In particular, p commutes with itself, so any permutation of $(a_1, \ldots, a_n) \models p^{(n)} | A$ is again realises $p^{(n)} | A$.

Examples

1. The generic type of $\mathcal{O}$ is stably dominated.

   Indeed, let $a \models p_{\mathcal{O}} \mid K$ and $K \subseteq L$. Then $\mathrm{St}_K(a) \mathop{\smile\!\!\!\!\!\!|}_K \mathrm{St}_K(L)$ just means that $\mathrm{res}(a) \notin k_L^{alg}$, forcing $a \models p_{\mathcal{O}} \mid L$.

2. The generic type of $\mathfrak{m}$ is not stably dominated.

   Indeed, we have $p_{\mathfrak{m}}(x) \otimes p_{\mathfrak{m}}(y) \vdash \mathrm{val}(x) < \mathrm{val}(y)$, whereas $p_{\mathfrak{m}}(y) \otimes p_{\mathfrak{m}}(x) \vdash \mathrm{val}(x) > \mathrm{val}(y)$.

3. On Γ_{∞}^{m}, only the realised types are stably dominated.

Characterisation of stably dominated types in ACVF

Definition

Let p be a definable type. We say p is **orthogonal to** Γ (and we denote this by $p \perp \Gamma$) if for every model M over which p is defined, letting $\bar{a} \models p \mid M$, one has $\Gamma(M) = \Gamma(M\bar{a})$.

Remark

Equivalently, in the defintion we may require the property to hold only for some model M over which p is defined.

Proposition

Let p be a definable type in ACVF. TFAE:

1. *p is stably dominated.*
2. *$p \perp \Gamma$.*
3. *p commutes with itself, i.e., $p(x) \otimes p(y) = p(y) \otimes p(x)$.*

Stably dominated types in ACVF: some closure properties

- **Realised types are stably dominated.**
- **Preservation under algebraic closure**:

  Suppose $\mathrm{tp}(\overline{a}/A)$ is stably dominated for some $A = \mathrm{acl}(A)$, and let $\overline{b} \in \mathrm{acl}(A\overline{a})$. Then $\mathrm{tp}(\overline{b}/A)$ is stably dominated, too.

  In particular, if p is stably dominated on X and $f : X \to Y$ is definable, then $f_*(p)$ is stably dominated on Y.
- **Transitivity**:

  If $\mathrm{tp}(\overline{a}/A)$ and $\mathrm{tp}(\overline{b}/A\overline{a})$ are both stably dominated, then $\mathrm{tp}(\overline{a}\overline{b}/A)$ is stably dominated, too.

  The converse of this is false in general. (See the examples below.)

Examples of stably dominated types in ACVF

- The generic type of a closed ball is stably dominated.
- The generic type of an open ball is not stably dominated.
- It follows that if $K \models \mathrm{ACVF}$ and $K \subseteq L = K(\overline{a})$ with $\mathrm{tr.deg}(L/K) = 1$, then $\mathrm{tp}(\overline{a}/K)$ is stably dominated iff $\mathrm{tr.deg}(k_L/k_K) = 1$.
- If $\mathrm{tr.deg}(L/K) = \mathrm{tr.deg}(k_L/k_K)$, then $\mathrm{tp}(\overline{a}/K)$ is stably dominated.
- There are more complicated stably dominated types: for every $n \geq 1$, there is $K \subseteq L = K(\overline{a})$ such that
  - $\mathrm{tr.deg}(L/K) = n$,
  - $\mathrm{tr.deg}(k_L/k_K) = 1$, and
  - $\mathrm{tp}(\overline{a}/K)$ is stably dominated.

Maximally complete models and metastability of ACVF

- A valued field K is maximally complete if it has no proper immediate extension.
- When working over a parameter set A, it is often useful to pass to a maximally complete $M \models \mathrm{ACVF}$ containing A, mainly due to the following important result.

Theorem (Haskell-Hrushovski-Macpherson)

Let M be a maximally complete model of ACVF, and let $\overline{a}$ be a tuple from $\mathbb{U}$. Then $\mathrm{tp}(\overline{a}/M, \Gamma(M\overline{a}))$ is stably dominated.

Remark

In abstract terms, the theorem states that ACVF is **metastable** *(over Γ), with metastability bases given by maximally complete models.*

Uniform definability of types

Fact

1. *Let T be stable and $\varphi(x, y)$ a formula. Then there is a formula $\chi(y, z)$ such that for every type $p(x)$ (over a model) there is b such that $d_p\varphi(y) = \chi(y, b)$.*
2. *The same result holds in ACVF if we restrict the conclusion to the collection of stably dominated types.*

Proof.

For every formula $\varphi(x, y)$ there is $n \geq 1$ such that whenever p is stably dominated and A-definable and $(a_0, \ldots, a_{2n}) \models p^{(2n+1)} \mid A$, then for any $b \in \mathbb{U}$, the **majority rule** holds, i.e.,

$$\varphi(x, b) \in p \text{ iff } \mathbb{U} \models \bigvee_{i_0 < \cdots < i_n} \varphi(a_{i_0}, b) \wedge \cdots \wedge \varphi(a_{i_n}, b). \quad \square$$

Prodefinable sets

Definition

A prodefinable set is a projective limit $D = \varprojlim_{i \in I} D_i$ of definable sets D_i, with def. transition functions $\pi_{i,j} : D_i \to D_j$ and I some small index set. (Identify $D(\mathbb{U})$ with a subset of $\prod D_i(\mathbb{U})$.)

We are only interested in **countable** index sets $\Rightarrow$ WMA $I = \mathbb{N}$.

Example

1. (**Type-definable sets**) If $D_i \subseteq \mathbb{U}^n$ are definable sets, $\bigcap_{i \in \mathbb{N}} D_i$ may be seen as a prodefinable set: WMA $D_{i+1} \subseteq D_i$, so the transition maps are given by inclusion.
2. $\mathbb{U}^\omega = \varprojlim_{i \in \mathbb{N}} \mathbb{U}^i$ is naturally a prodefinable set.

Some notions in the prodefinable setting

Let $D = \varprojlim_{i \in I} D_i$ and $E = \varprojlim_{j \in J} E_j$ be prodefinable.

- There is a natural notion of a **prodefinable map** $f : D \to E$ [f is given by a compatible system of maps $f_j : D \to E_j$, each f_j factoring through some component $D_{i(j)}$]
- D is called **strict prodefinable** if it can be written as a prodefinable set with surjective transition functions.
- D is called **iso-definable** if it is in prodefinable bijection with a definable set.
- $X \subseteq D$ is called **relatively definable** if there is $i \in I$ and $X_i \subseteq D_i$ definable such that $X = \pi_i^{-1}(X_i)$.

The set of definable types as a prodefinable set (T stable)

- Assume T is stable with EI (e.g. $T = \mathrm{ACF}_p$)
- For any $\varphi(x, y)$ fix $\chi_\varphi(y, z)$ s.t. for any definable type $p(x)$ we have $d_p\varphi(y) = \chi_\varphi(y, b)$ for some $b = \ulcorner d_p\varphi \urcorner$.
- For X definable, let $S_{def,X}(A)$ be the A-definable types on X.

Proposition

1. *There is a prodefinable set D such that $S_{def,X}(A) = D(A)$ naturally. (Identify $p \mid \mathbb{U}$ with the tuple $(\ulcorner d_p\varphi \urcorner)_\varphi$).*
2. *If $Y \subseteq X$ is definable, $S_{def,Y}$ is relatively definable in $S_{def,X}$.*
3. *The subset of $S_{def,X}$ corresponding to the set of realised types is relatively definable and isodefinable. (It is $\cong X(\mathbb{U})$.)*

Strict pro-definability and nfcp

Problem

Let $D_{\varphi,\chi} = \{b \in U \mid \chi(y,b)$ is the φ-definition of some type$\}$. Then $D_{\varphi,\chi}$ is not always definable.

Fact

In ACF, all $D_{\varphi,\chi}$ are definable. More generally, for a stable theory T this is the case iff T is **nfcp**.

Corollary

1. *If T is stable and nfcp (e.g. $T = \mathrm{ACF}$), then $S_{def,X}$ is strict pro-definable.*
2. *If C is a curve definable over $K \models \mathrm{ACF}$, then $S_{def,C}$ is iso-definable.*
3. *$S_{def,\mathbb{A}^2}$ is not iso-definable in ACF: the generic types of the curves given by $y = x^n$ cannot be seperated by finitely many φ-types.*

The set of stably dominated types as a prodefinable set

For X an A-definable set in ACVF, we denote by $\widehat{X}(A)$ the set of A-definable stably dominated types on X.

Theorem

Let X be C-definable. There exists a strict C-prodefinable set D such that for every $A \supseteq C$, we have a canonical identification $\widehat{X}(A) = D(A)$.

Once the theorem is established, we will denote by $\widehat{X}$ the prodefinable set representing it.

Proof of the theorem.

For notational simplicity, we will assume $C = \emptyset$.

- Let $f : X \to \Gamma_\infty$ be definable (with parameters) and let $p \in \widehat{X}(\mathbb{U})$. Then $f_*(p)$ is stably dominated on Γ_∞, so is a realised type $x = \gamma$. We will denote this by $f(p) = \gamma$.
- Now let $f : W \times X \to \Gamma_\infty$ be $\emptyset$-definable, $f_w := f(w, -)$. Then there is a set S and a function $g : W \times S \to \Gamma_\infty$, both $\emptyset$-definable, such that for every $p \in \widehat{X}(\mathbb{U})$, the function

$$f_p : W \to \Gamma_\infty,\ w \mapsto f_w(p)$$

  is equal to $g_s = g(s, -)$ for a unique $s \in S$.

  This follows from
  - uniform definability of types for stably dominated types, and
  - elimination of imaginaries in ACVF (in $\mathcal{L}_\mathcal{G}$).

End of the proof

Choose an enumeration $f_i : W_i \times X \to \Gamma_\infty$ ($i \in \mathbb{N}$) of the functions as above (with corresponding $g_i : W_i \times S_i \to \Gamma_\infty$).

Then $p \mapsto c(p) := \{(s_i)_{i\in\mathbb{N}} \mid f_{i,p} = g_{i,s_i} \text{ for all } i\}$ defines an injection of $\widehat{X}$ into $\prod_i S_i$.

The strict prodefinable set we are aiming for is $D = c(\widehat{X})$.
Let $I \subseteq \mathbb{N}$ be finite and $\pi_I(D) = D_I \subseteq \prod_{i\in I} S_i$. We finish by the following two facts:

- D_I is type-definable. (This gives prodefinability of D.)
  [This is basically compactness and QE.]
- D_I is a union of definable sets.
  [This uses $\mathrm{St}_A = \mathrm{VS}_{\mathbf{k},A}$, and these are 'uniformly' nfcp.]

$\Rightarrow$ the D_I are definable, proving strict prodefinability of D. □

Some definability properties in $\widehat{X}$

- **Functoriality**:

  For any definable $f : X \to Y$, we get a prodefinable map $\widehat{f} : \widehat{X} \to \widehat{Y}$.

- **Passage to definable subsets**:

  If Y is a definable subset of X, then $\widehat{Y} \subseteq \widehat{X}$ is a relatively definable subset.

- **Simple points**:

  The set of realised types in $\widehat{X}$, in natural bijection with $X(\mathbb{U})$, is iso-definable and relatively definable in $\widehat{X}$.

  Elements of $\widehat{X}$ corresponding to realised types will be called **simple** points.

Isodefinability in the case of curves

Theorem

Let C be an algebraic curve. Then $\widehat{C}$ is iso-definable.

Proof.

- WMA C is smooth and projective, $C \subseteq \mathbb{P}^n$. Let $g = genus(C)$.
- In $K(\mathbb{P}^1) = K(X)$, any element is a product of linear polynomials in X. The following consequence of Riemann-Roch gives a generalisation of this to arbitrary genus:

  There exists an N ($N = 2g + 1$ is enough) s.t. any non-zero $f \in K(C)$ is a product of functions of the form $(g/h) \restriction_C$, where $g, h \in K[X_0, \ldots, X_n]$ are homogeneous of degree N.
- Thus any valuation on $K(C)$ is determined by its values on a definable family of polynomials, proving iso-definability. □

Isodefinability in the case of curves (continued)

From now on, we will write $\mathcal{B}^{cl}$ for the set of closed balls including singletons (closed balls of radius ∞).

Examples

1. If $C = \mathbb{A}^1$, the isodefinability of $\widehat{C}$ is clear, as then $\widehat{\mathbb{A}^1} = \mathcal{B}^{cl}$ (which is a definable set).
2. $\widehat{\mathcal{O}^2}$ is not isodefinable. Indeed, let $p_{\mathcal{O}}$ be the generic of $\mathcal{O}$, and $p_n(x, y) \in \widehat{\mathcal{O}^2}$ be given by $p_{\mathcal{O}}(x) \cup \{y = x^n\}$.

   No definable family of functions to Γ_∞ allows to separate all the p_n's, as $\mathrm{val}(f(p_n)) = \mathrm{val}(f(p_{\mathcal{O}}(x) \otimes p_{\mathcal{O}}(y)))$ for all $f \in K[X, Y]$ of degree $< n$.

Remark

For $X \subseteq K^n$ definable, $\widehat{X}$ is iso-definable iff $\dim(X) \leq 1$.

(Here, $\dim(X)$ denotes the algebraic dimension of X^{Zar}.)

Prodefinable topological spaces

Definition

Let X be (pro-)definable over A.
A topology $\mathcal{T}$ on $X(\mathbb{U})$ is said to be A-**definable** if

- there are A-definable families $\mathcal{W}^i = (W^i_b)_{b\in\mathbb{U}}$ (for $i \in I$) of (relatively) definable subsets of X such that
- the topology on $X(\mathbb{U})$ is generated by the sets (W^i_b), where $i \in I$ and $b \in \mathbb{U}$.

We call $(X, \mathcal{T})$ a **(pro-)definable space**.

Remark

1. *Given a (pro-)definable space $(X, \mathcal{T})$ (over A) and $A \subseteq M \preccurlyeq \mathbb{U}$, the M-definable open sets from $\mathcal{T}$ define a topology on $X(M)$.*
2. *The inclusion $X(M) \subseteq X(\mathbb{U})$ is in general not continuous.*

Examples of definable topologies

1. If M is o-minimal, then M^n equipped with the product of the order topology is a definable space.
2. Let V be an algebraic variety over $K \models \mathrm{ACVF}$. Then the valuation topology on $V(K)$ is definable.
3. The Zariski topology on $V(K)$ is a definable topology.

Remark

- *The topologies in examples (1) and (2) are* **definably generated**, *in the sense that a single family of definable open sets generates the topology. (There is even a definable basis of the topology in both cases.)*
- *The Zariski topology in (3) is not definably generated, unless* $\dim(V) \leq 1$.

$\widehat{V}$ as a prodefinable space

Given an algebraic variety V defined over $K \models \mathrm{ACVF}$, we will define a definable topology on $\widehat{V}$, turning it into a prodefinable space, the **Hrushovski-Loeser space** associated to V.

The construction of the topology is done in several steps:

- We will give an explicit construction in the case $V = \mathbb{A}^n$.
- If V is affine, $V \subseteq \mathbb{A}^n$ a closed embedding, we give $\widehat{V}$ the subspace topology inside $\widehat{\mathbb{A}^n}$.
- The case of an arbitrary V done by gluing affine pieces: if $V = \bigcup U_i$ is an open affine cover, $\widehat{V} = \bigcup \widehat{U}_i$ is an open cover.
- Let $X \subseteq V$ be a definable subset of the variety V. Then we give $\widehat{X}$ the subspace topology inside $\widehat{V}$.

  Subsets of $\widehat{V}$ of the form $\widehat{X}$ will be called **semi-algebraic**.

The topology on $\widehat{\mathbb{A}^n}$

Recall that any definable function $f : X \to \Gamma_\infty$ canonically extends to a map $f : \widehat{X} \to \Gamma_\infty$ (given by the composition $\widehat{X} \xrightarrow{\widehat{f}} \widehat{\Gamma_\infty} \xrightarrow{=} \Gamma_\infty$).

Definition

We endow $\widehat{\mathbb{A}^n}(\mathbb{U})$ with the topology generated by the (so-called *pre-basic open*) sets of the form

$$\{a \in \widehat{\mathbb{A}^n} \mid \mathrm{val}(F(a) < \gamma\} \text{ or } \{a \in \widehat{\mathbb{A}^n} \mid \mathrm{val}(F(a) > \gamma\},$$

where $F \in \mathbb{U}[x_1, \ldots, x_n]$ and $\gamma \in \Gamma(\mathbb{U})$.

Remark

1. *The topology is the coarsest one such that for all polynomials F, the map $\mathrm{val} \circ F : \widehat{\mathbb{A}^n} \to \Gamma_\infty$ is continuous. (Here, Γ_∞ is considered with the order topology.)*
2. *It has a basis of open semialgebraic sets.*

Proposition

The topology on $\widehat{V}$ is pro-definable, over the same parameters over which V is defined.

Proof.

- By our construction, it is enough to show the result for $V = \mathbb{A}^n$.
- For any d, the pre-basic open sets defined by polynomials of degree $\leq d$ form a definable family of relatively definable subsets of $\widehat{\mathbb{A}^n}$. □

Relationship with the order topology

- For a closed ball b, let p_b be the generic type of b. The map

$$\gamma : \Gamma_\infty^m \to \widehat{\mathbb{A}^m}, (t_1, \ldots, t_m) \mapsto p_{B_{\geq t_1}(0)} \otimes \cdots \otimes p_{B_{\geq t_m}(0)}$$

  is a definable homeomorphism onto its image, where Γ_∞^m is endowed with the (product of the) order topology.

- Let $f = \mathrm{id} \times (\mathrm{val}, \ldots, \mathrm{val}) : V \times \mathbb{A}^m \to V \times \Gamma_\infty^m$.
  On $\widehat{V \times \Gamma_\infty^m}$ we put the topology induced by $\widehat{f}$, i.e.
  $U \subseteq \widehat{V \times \Gamma_\infty^m}$ is open iff $\widehat{f}^{-1}(U)$ is open in $\widehat{V \times \mathbb{A}^m}$.

Fact

$\widehat{\Gamma_\infty^m} = \Gamma_\infty^m$. Moreover, the map $\widehat{V \times \Gamma_\infty^m} \to \widehat{V} \times \widehat{\Gamma_\infty^m} = \widehat{V} \times \Gamma_\infty^m$ is a homeomorphism, where Γ_∞ is endowed with the order topology.

Example (The topology on $\widehat{\mathbb{A}^1}$)

- Recall that $\widehat{\mathbb{A}^1} = \mathcal{B}^{cl}$ as a set.
- A semialgebraic subset $\widehat{X} \subseteq \widehat{\mathbb{A}^1}$ is open iff X is a finite union of sets of the form $\Omega \setminus \bigcup_{i=1}^{n} F_i$, where
  - Ω is an open ball or the whole field K;
  - the F_i are closed sub-balls of Ω.
- $\widehat{\mathfrak{m}}$ and $\widehat{\mathfrak{m}} \setminus \{0\}$ are open, with closure equal to $\widehat{\mathfrak{m}} \cup \{p_{\mathcal{O}}\}$, a definable closed set which is not semi-algebraic.
- $\{p_b \mid \mathrm{rad}(b) > \alpha\}$ $(\alpha \in \Gamma)$ is def. open and non semi-algebraic.
- The topology is definably generated by the family $\{\widehat{\Omega \setminus F}\}_{\Omega,F}$.
- There is no definable basis for the topology.

Fact

For any curve C, the topology on $\widehat{C}$ is definably generated.

[This follows from the proof of iso-definability of $\widehat{C}$.]

First properties of the topological space $\widehat{V}$

Fact

Let V be an algebraic variety defined over $M \models \mathrm{ACVF}$.

1. *The topologicy on $\widehat{V}(M)$ is Hausdorff.*
2. *The subset $V(M)$ of simple points is dense in $\widehat{V}(M)$.*
3. *The induced topology on $V(M)$ is the valuation topology.*

Proof.

We will assume that V is affine, say $V \subseteq \mathbb{A}^n$.

For (1), let $p, q \in \widehat{V}(M)$ with $p \neq q$. There is $F(\overline{x}) \in K[\overline{x}]$ such that $\mathrm{val}(F(p)) \neq \mathrm{val}(F((q))$, say $\mathrm{val}(F(p)) < \alpha < \mathrm{val}(F((q))$, where $\alpha \in \Gamma(M)$. Then $\mathrm{val}(F(\overline{x})) < \alpha$ and $\mathrm{val}(F(\overline{x})) > \alpha$ define disjoint open sets in $\widehat{V}$, one containing p, the other containing q.

(2) and (3) follows from the fact that there is a basis of the topology given by semialgebraic open sets. □

The v+g-topology

- Let V be a variety and $X \subseteq V$ definable. We say
  - X is **v-open** (in V) if it is open for the valuation topology;
  - X is **g-open** (in V) if it is given (inside V) by a positive Boolean combination of *Zariski constructible* sets and sets defined by *strict valuation inequalities* $\mathrm{val}(F(\overline{x})) < \mathrm{val}(G(\overline{x}))$;
  - X **v+g-open** (in V) if it is v-open and g-open.
- We say $X \subseteq V \times \Gamma_\infty^m$ is v-open iff its pullback to $V \times \mathbb{A}^m$ is. (Similarly for g-open and v+g-open.)

Remark

The g-open and the v+g-open sets do not give rise to a definable topology. Indeed, $\mathcal{O}$ is not g-open, but $\mathcal{O} = \bigcup_{a \in \mathcal{O}} a + \mathfrak{m}$, so it is a definable union of v+g-open sets.

Why consider the v-topology and the g-topology?

- With the two topologies (v and g), one may separate continuity issues related to very different phenomena in Γ_∞, namely
  - the **behaviour near** ∞ (captured by the v-topology) and
  - the **behaviour near** $0 \in \Gamma$ (captured by the g-topology).
- It is e.g. easier to check continuity separately.
- v+g-topology on $V \longleftrightarrow$ topology on $\widehat{V}$ (see on later slides)

Exercise

- The v-topology on Γ_∞ is discrete on Γ, and a basis of open neighbourhoods at ∞ is given by $\{(\alpha, \infty], \alpha \in \Gamma\}$.
- The g-topology on Γ_∞ corresponds to the order topology on Γ, with ∞ isolated.
- Thus, the v+g-topology on Γ_∞ is the order topology.

Limits of definable types in (pro-)definable spaces

Definition

Let $p(x)$ a definable type on a pro-definable space X.

We say $a \in X$ is a **limit** of p if $p(x) \vdash x \in W$ for every $\mathbb{U}$-definable neighbourhood W of a.

Remark

If X is Hausdorff space, then limits are unique (if they exist), and we write $a = \lim(p)$.

Example

Let M be an o-minimal structure and $\alpha \in M$. Then $\alpha = \lim(\alpha^+)$.

Describing the closure with limits of definable types

Proposition

Let X be prodefinable subset of $\widehat{V \times \Gamma_\infty^m}$.

1. *If X is closed, then it is closed under limits of definable types, i.e. if p is a definable type on X such that $\lim(p)$ exists in $\widehat{V \times \Gamma_\infty^m}$, then $\lim(p) \in X$.*
2. *If $a \in \mathrm{cl}(X)$, there is a def. type p on X such that $a = \lim(p)$. Thus, X closed under limits of definable types $\Rightarrow$ X closed.*

Example

Recall that $\mathrm{cl}(\widehat{\mathfrak{m} \setminus \{0\}}) = \widehat{\mathfrak{m}} \cup \{p_{\mathcal{O}}\}$.

- Let q_{0^+} be the (definable) type giving the generic type in the closed ball of radius $\epsilon \models 0^+$ around 0. Then $p_{\mathcal{O}} = \lim(q_{0^+})$.
- Similarly, $0 \widehat{=} B_{\geq\infty}(0) = \lim(q_{\infty^-})$.

Definable compactness

Definition

A (pro-)definable space X is said to be **definably compact** if every definable type on X has a limit in X.

Remark

In an o-minimal structure M, this notion is equivalent to the usual one, i.e. a definable subset $X \subseteq M^n$ is definably compact iff it is closed and bounded.

Lemma (The key to the notion of definable compactness)

Let $f : X \to Y$ be a surjective (pro-)definable map between (pro-)definable sets (in ACVF). Then the induced maps $f_{def} : S_{def,X} \to S_{def,Y}$ and $\widehat{f} : \widehat{X} \to \widehat{Y}$, are surjective, too.

Corollary

Assume $f : \widehat{V} \times \Gamma_\infty^m \to \widehat{W} \times \Gamma_\infty^n$ is definable and continuous, and $X \subseteq \widehat{V} \times \Gamma_\infty^m$ is a pro-definable and definably compact subset. Then $f(X)$ is definably compact.

Proof of the corollary.

- By the lemma, any definable type p on $f(X)$ is of the form $f_* q = f_{def}(q)$ for some definable type q on X.
- As X is definably compact, there is $a \in X$ with $\lim(q) = a$.
- By continuity of f, we get $\lim(p) = f(a)$. □

Bounded subsets of algebraic varieties

Definition

- Let $V \subseteq \mathbb{A}^m$ be a closed subvariety. Wa say a definable set $X \subseteq V$ is **bounded** (in V) if $X \subseteq c\mathcal{O}^m$ for some $c \in K$.
- For general V, $X \subseteq V$ is called bounded (in V) if there is an open affine cover $V = \bigcup_{i=1}^n U_i$ and $X_i \subseteq U_i$ with X_i bounded in U_i such that $X = \bigcup_{i=1}^n X_i$.
- $X \subseteq V \times \Gamma_\infty^m$ is said to be bounded (in $V \times \Gamma_\infty^m$) if its pullback to $V \times \mathbb{A}^m$ is bounded in $V \times \mathbb{A}^m$.
- Finally, we say that a pro-definable subset $X \subseteq \widehat{V}$ is bounded (in $\widehat{V}$) if there is $W \subseteq V$ bounded such that $X \subseteq \widehat{W}$.

Fact

The notion is well-defined (i.e. independent of the closed embedding into affine space and of the choice of an open affine cover).

Bounded subsets of algebraic varieties (continued)

Examples

1. $X \subseteq \Gamma_\infty$ is bounded iff $X \subseteq [\gamma, \infty]$ for some $\gamma \in \Gamma$.
2. $\mathbb{P}^n$ is bounded in itself, so every $X \subseteq \mathbb{P}^n$ is bounded.
   Indeed, if $\mathbb{A}^n \cong U_i$ is the affine chart given by $x_i \neq 0$ and $U_i(\mathcal{O}) \subseteq U_i$ corresponds to $\mathcal{O}^n \subseteq \mathbb{A}^n$, then we may write $\mathbb{P}^n = \bigcup_{i=0}^n U_i(\mathcal{O})$.
3. $\mathbb{A}^1$ is bounded in $\mathbb{P}^1$ and unbounded in itself, so the notion depends on the ambient variety.

A characterisation result for definable compactness

Theorem

Let $X \subseteq \widehat{V \times \Gamma_\infty^m}$ be pro-definable. TFAE:

1. *X is definably compact.*
2. *X is closed and bounded.*

To illustrate the methods, we will prove that if $X \subseteq \widehat{V \times \Gamma_\infty^m}$ is bounded, then any definable type on X has a limit in $\widehat{V \times \Gamma_\infty^m}$.

Corollary

Let $W \subseteq V \times \Gamma_\infty^m$.

1. *$\widehat{W}$ is closed in $\widehat{V \times \Gamma_\infty^m}$ iff W is v+g-closed in $V \times \Gamma_\infty^m$.*
2. *$\widehat{W}$ is definably compact iff W is a v+g-closed and bounded subset of $V \times \Gamma_\infty^m$.*

Some further applications of the characterisation result

The results below are analogous to the complex situation.

Corollary

A variety V is complete iff $\widehat{V}$ is definably compact.

Proof.

- By Chow's lemma, if V is complete there is $f : V' \to V$ surjective with V' projective. This proves one direction.
- For the other direction, use that every variety is an open Zariski dense subvariety of a complete variety. □

Corollary

If $f : V \to W$ is a proper map between algebraic varieties, then $\widehat{f} : \widehat{V} \to \widehat{W}$ as well as $\widehat{f} \times \mathrm{id} : \widehat{V} \times \Gamma_\infty^m \to \widehat{W} \times \Gamma_\infty^m$ are closed maps.

Proof that definable types on bounded sets have limits

Lemma

Let p be a definable type on a bounded subset $X \subseteq \widehat{V \times \Gamma_\infty^m}$. Then $\lim(p)$ *exists in* $\widehat{V \times \Gamma_\infty^m}$.

Proof.

- First we reduce to the case where $V = \mathbb{A}^n$ and $m = 0$.
- Let $K \models \mathrm{ACVF}$ be maximally complete, with p K-definable, $d \models p \mid K$ and $a \models p_d \mid Kd$, where p_d is the type coded by d.
- As $p_d \perp \Gamma$, we have $\Gamma_K \subseteq \Gamma(K(d)) = \Gamma(K(d,a)) =: \Delta$.

  Let $\Delta_0 := \{\delta \in \Delta \mid \exists \gamma \in \Gamma_K : \gamma < \delta\}$.
- p definable $\Rightarrow$ for $\delta \in \Delta_0$, $\mathrm{tp}(\delta/\Gamma_K)$ is definable and has a limit in $\Gamma_K \cup \{\infty\}$.

End of the proof

(Recall: $\Delta_0 := \{\delta \in \Delta \mid \exists \gamma \in \Gamma_K : \gamma < \delta\}$)

- We get a retraction $\pi : \Delta_0 \to \Gamma_K \cup \{\infty\}$ preserving $\leq$ and $+$.
- $\mathcal{O}' := \{b \in K(a) \mid \mathrm{val}(b) \in \Delta_0\}$ is a valuation ring on $K(a)$.
- As $K \subseteq \mathcal{O}'$, putting $\widetilde{\mathrm{val}}(x + \mathfrak{m}') := \pi(\mathrm{val}(x))$, we get a valued field extension $\tilde{K} = \mathcal{O}'/\mathfrak{m}' \supseteq K$, with $\Gamma_{\tilde{K}} = \Gamma_K$.
- The coordinates of a lie in $\mathcal{O}'$, by the boundedness of X.
- Consider the tuple $\tilde{a} := a + \mathfrak{m}' \in K'$.
  - Then $r = \mathrm{tp}(a'/K)$ is stably dominated as $\Gamma(Ka') = \Gamma(K)$ and K is maximally complete.
  - One checks that $r = \lim(p)$. (Indeed, one shows $f(r) = \lim(f_*(p))$ for every $f = \mathrm{val} \circ F$, where $F \in K[\overline{x}]$.) □

Γ-internal subsets of $\widehat{V}$

Convention

From now on, all varieties are assumed to be quasi-projective.

Definition

A subset $Z \subseteq \widehat{V \times \Gamma_\infty^m}$ is called Γ-**internal** if

- Z is iso-definable and
- there is a surjective definable $f : D \subseteq \Gamma_\infty^n \twoheadrightarrow Z$.

Remark

If we drop in the definition the iso-definability requirement, we get the weaker notion called Γ-parametrisability.

Fact

Let $f : C \to C'$ be a finite morphism between algebraic curves. Assume that $Z \subseteq \widehat{C}$ is Γ-internal. Then $\hat{f}^{-1}(Z)$ is Γ-internal.

Topological witness for Γ-internality

Proposition

Let $Z \subseteq \widehat{V \times \Gamma_\infty^m}$ be Γ-internal. Then there is an injective continuous definable map $f : Z \hookrightarrow \Gamma_\infty^n$ for some n. If Z is definably compact, such an f is a homeomorphism.

The question is more delicate if one wants to control the parameters needed to define f. Here is the best one can do:

Proposition

Suppose that in the above, both V and Z are A-definable, where $A \subseteq \mathbf{VF} \cup \Gamma$. Then there is a finite A-definable set w and an injective continuous A-definable map $f : Z \hookrightarrow \Gamma_\infty^w$.

Example

Let $A = \mathbb{Q} \subseteq \mathbf{VF}$, V given by $X^2 - 2 = 0$. Then $\widehat{V}$ is Γ-internal, with a non-trivial Galois action, so cannot be $\mathbb{Q}$-embedded into Γ_∞^n.

Generalised intervals

We say that $\mathrm{I} = [o_{\mathrm{I}}, e_{\mathrm{I}}]$ is a **generalised closed interval** in Γ_∞ if it is obtained by concatenating a finite number of closed intervals $\mathrm{I}_1, \ldots, \mathrm{I}_n$ in Γ_∞, where $<_{\mathrm{I}_i}$ is either given by $<_{\Gamma_\infty}$ or by $>_{\Gamma_\infty}$.

Remark

- *The absence of the multiplication in* Γ_∞ *makes it necessary to consider generalised intervals.*
- *E.g., there is a definable path* $\gamma : \mathrm{I} \to \widehat{\mathbb{P}^1}$ *with* $\gamma(o_{\mathrm{I}}) = 0$ *and* $\gamma(e_{\mathrm{I}}) = 1$*, but only if we allow generalised intervals in the definition of a path.*

Definable homotopies and strong deformation retractions

Definition

Let $\mathrm{I} = [o_{\mathrm{I}}, e_{\mathrm{I}}]$ be a generalised interval in Γ_∞ and let $X \subseteq V \times \Gamma_\infty^m$, $Y \subseteq W \times \Gamma_\infty$ be definable sets.

1. A continuous pro-definable map $H : \mathrm{I} \times \widehat{X} \to \widehat{Y}$ is called a **definable homotopy** between the maps $H_o, H_e : \widehat{X} \to \widehat{Y}$, where H_o corresponds to $H \restriction_{\{o_{\mathrm{I}}\} \times \widehat{X}}$ (similarly for H_e).
2. We say that the definable homotopy $H : \mathrm{I} \times \widehat{X} \to \widehat{X}$ is a **strong deformation retraction** onto the set $\Sigma \subseteq \widehat{X}$ if
   - $H_0 = \mathrm{id}_{\widehat{X}}$,
   - $H \restriction_{\mathrm{I} \times \Sigma} = \mathrm{id}_{\mathrm{I} \times \Sigma}$,
   - $H_e(\widehat{X}) \subseteq \Sigma$, and
   - $H_e(H(t, a)) = H_e(a)$ for all $(t, a) \in \mathrm{I} \times \widehat{X}$.

We added the last condition, as it is satisfied by all the retractions we will consider.

The standard homotopy on $\widehat{\mathbb{P}^1}$

- We represent $\mathbb{P}^1(\mathbb{U})$ as the union of two copies of $\mathcal{O}(\mathbb{U})$, according to the two affine charts w.r.t. u and $\frac{1}{u}$, respectively.
- In this way, unambiguously, any element of $\widehat{\mathbb{P}^1}$ corresponds to the generic type $p_{B_{\geq s}(a)}$ of a closed ball of val. radius $s \geq 0$.

Definition

The **standard homotopy** on $\widehat{\mathbb{P}^1}$ is defined as follows:

$$\psi : [0, \infty] \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1},\ (t, p_{B_{\geq s}(a)}) \mapsto p_{B_{\geq \min(s,t)}(a)}$$

Lemma

The map ψ is continuous. Viewing $[0, \infty]$ as a (generalised) interval with $o_{\mathrm{I}} = \infty$ and $e_{\mathrm{I}} = 0$, ψ is a strong deformation retraction of $\widehat{\mathbb{P}^1}$ onto the singleton set $\{p_{\mathcal{O}}\}$.

A variant: the standard homotopy with stopping time D

- $\mathbb{P}^1(\mathbb{U})$ has a tree-like structure: any two elements $a, b \in \mathbb{P}^1(\mathbb{U})$ are the endpoints of a unique *segment*, i.e. a subset of $\widehat{\mathbb{P}^1}$ definably homeomorphic to a (generalised) interval in Γ_∞.
- Given $D \subseteq \mathbb{P}^1$ finite, let C_D be the convex hull of $D \cup \{p_{\mathcal{O}}\}$ in $\widehat{\mathbb{P}^1}$, i.e. the image of $[0, \infty] \times (D \cup \{p_{\mathcal{O}}\})$ under ψ.
- C_D is closed in $\widehat{\mathbb{P}^1}$ and Γ-internal, and the map $\tau : \widehat{\mathbb{P}^1} \to \Gamma_\infty$, $\tau(b) := \max\{t \in [0, \infty] \mid \psi(t, b) \in C_D\}$ is continuous.

Lemma

Consider the **standard homotopy with stopping time** D*,*

$$\psi_D : [0, \infty] \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1}\ (t, b) \mapsto \psi(\max(\tau(b), t), b).$$

Then ψ_D defines a strong deformation retraction of $\widehat{\mathbb{P}^1}$ onto C_D.

A strong deformation retraction for curves

Theorem

Let C be an algebraic curve. Then there is a strong deformation retraction $H : [0, \infty] \times \widehat{C} \to \widehat{C}$ onto a Γ-internal subset $\Sigma \subseteq \widehat{C}$.

Sketch of the proof.

- WMA C is projective.
- Choose $f : C \to \mathbb{P}^1$ finite and generically étale.
- Idea: one shows that there is $D \subseteq \mathbb{P}^1$ finite such that $\psi_D : [0, \infty] \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1}$ 'lifts' (uniquely) to a strong deformation retraction $H : [0, \infty] \times \widehat{C} \to \widehat{C}$, i.e., such that $H \circ \widehat{f} = \psi_D \circ (\mathrm{id} \times \widehat{f})$ holds.

Outward paths on finite covers of $\mathbb{A}^1$

Definition

- A **standard outward path on** $\widehat{\mathbb{A}^1}$ **starting at** $a = p_{B_{\geq s}(c)}$ is given by $\gamma : (r, s] \to \widehat{\mathbb{A}^1}$ (for some $r < s$) such that $\gamma(t) = p_{B_{\geq t}(c)}$.
- Let $f : C \to \mathbb{A}^1$ be a finite map. An **outward path on** $\widehat{C}$ **starting at** $x \in \widehat{C}$ (with respect to f) is a continuous definable map $\gamma : (r, s] \to \widehat{C}$ for some $r < s$ such that
  - $\gamma(s) = x$ and
  - $\widehat{f} \circ \gamma$ is a standard outward path on $\widehat{\mathbb{A}^1}$.

Lemma

Let $f : C \to \mathbb{A}^1$ be a finite map. Then, for every $x \in \widehat{C}$, there exists at least one and at most $\deg(f)$ *many outward paths starting at x (with respect to f).*

Finiteness of outward branching points

- Let $f : C \to \mathbb{A}^1$ be a finite map, $d = \deg(f)$.
- Note that for all $x \in \widehat{\mathbb{A}^1}$, we have $\mid \widehat{f}^{-1}(x) \mid \leq d$.
- We say $y \in \widehat{C}$ is **outward branching** (for f) if there is more than one outward path on $\widehat{C}$ starting at y. In this case, we also say that $\widehat{f}(y) \in \widehat{\mathbb{A}^1}$ is outward branching.

Key lemma

The set of outward branching points (for f) is finite.

End of the proof

Suppose $f : C \to \mathbb{P}^1$ is finite and generically étale.

By the key lemma, there is $D \subseteq \mathbb{P}^1$ finite such that

- f is étale above $\mathbb{P}^1 \setminus D$;
- C_D contains all outward branching points, with respect to the maps restricted to the two standard affine charts.

Lemma

Under the above assumptions, the map $\psi_D : [0, \infty] \times \widehat{\mathbb{P}^1} \to \widehat{\mathbb{P}^1}$ lifts (uniquely) to a strong deformation retraction $H : [0, \infty] \times \widehat{C} \to \widehat{C}$.

Example

Consider the elliptic curve E given by the affine equation $y^2 = x(x-1)(x-\lambda)$, where $\mathrm{val}(\lambda) > 0$ (in char $\neq 2$).
Let $f : E \to \mathbb{P}^1$ be the map to the x-coordinate.

- f is ramified at $0, 1, \lambda$ and ∞.
- Using Hensel's lemma, one sees that the fiber of $\widehat{f}$ above $x \in \widehat{\mathbb{A}^1}$ has two elements iff x is neither in the segment joining 0 and λ, nor in the one joining 1 and ∞.
- Thus, for $B = B_{\geq \mathrm{val}(\lambda}(0)$, the point p_B is the unique outward branching point on the affine line corresponding to $x \neq \infty$.
- On the affine line corresponding to $x \neq 0$, $p_{\mathcal{O}}$ is the only outward branching point.
- We may thus take $D = \{0, \lambda, 1, \infty\}$.
- If H is the unique lift of ψ_D, then H defines a retraction of $\widehat{E}$ onto a subset of $\widehat{E}$ which is homotopic to a circle.

Definable connectedness

Definition

Let V be an algebraic variety and $Z \subseteq \widehat{V}$ strict pro-definable.

- Z is called **definably connected** if it contains no proper non-empty clopen strict pro-definable subset.
- Z is called **definably path-connected** if any two points $z, z' \in Z$ are connected by a definable path.

The following lemma is easy.

Lemma

1. *Z definably path-connected $\Rightarrow$ Z definably connected*
2. *For $X \subseteq V$ definable, $\widehat{X}$ is definably connected iff X does not contain any proper non-empty v+g-clopen definable subset.*
3. *If $\widehat{V}$ is definably connected, then V is Zariski-connected.*

GAGA for connected components

- For $X \subseteq V$ definable, we say $\widehat{X}$ has **finitely many connected components** if X admits a finite definable partition into v+g-clopen subsets Y_i such that $\widehat{Y_i}$ is definably connected.
- The $\widehat{Y_i}$ are then called the **connected components** of $\widehat{X}$.

Theorem

Let V be an algebraic variety.

- *$\widehat{V}$ is definably connected iff V is Zariski connected.*
- *$\widehat{V}$ has finitely many connected components, which are of the form $\widehat{W}$, for W a Zariski connected component of V.*

Proof of the theorem: reduction to smooth projective curves

Lemma

Let V be a smooth variety and $U \subseteq V$ an open Zariski-dense subvariety of V. Then $\widehat{V}$ has finitely many connected components if and only if $\widehat{U}$ does. Moreover, in this case there is a bijection between the two sets of connected components.

We assume the lemma (which will be used several times).

- WMA V is Zariski-connected.
- WMA V is irreducible.
- Any two points $v \neq v' \in V$ are contained in an irreducible curve $C \subseteq V$. This uses Chow's lemma and Bertini's theorem.
  $\Rightarrow$ WMA $V = C$ is an **irreducible curve**.
- WMA C is **projective** (by the lemma) and **smooth** (passing to the normalisation $\tilde{C} \twoheadrightarrow C$)

The case of a smooth projective curve C

We have already seen:

$\widehat{C}$ retracts to a Γ-internal (PL) subspace $S \subseteq \widehat{C}$

$\Rightarrow$ $\widehat{C}$ has finitely many conn. components (all path-connected)

- If $g(C) = 0$, $C \cong \mathbb{P}^1$, so $\widehat{C}$ is contractible (thus connected).
- If $g(C) = 1$, $C \cong E$, where E is an elliptic curve.
  - $(E(\mathbb{U}), +)$ acts on $\widehat{E}(\mathbb{U})$ by definable homeomorphisms;
  - this action is transitive on simple points (which are dense).

  $\Rightarrow$ $E(\mathbb{U})$ acts transitively on the (finite!) set of connected components of $\widehat{E}$.

  $\Rightarrow$ $\widehat{E}$ is connected, since $E(\mathbb{U})$ is divisible.

The case of a smooth projective curve C, with $g(C) \geq 2$.

- Let $\widehat{C_0}, \ldots, \widehat{C_{n-1}}$ be the connected components of $\widehat{C}$.
- For $I = (i_1, \ldots, i_g) \in n^g$, $C_I := C_{i_1} \times \cdots \times C_{i_g}$ is a v+g-clopen subset of C^g, and $\widehat{C_I}$ is definably connected.
- Thus, $\widehat{C^g}$ has n^g connected components. If $n \geq 2$, $\widehat{C^g}$ as well as $\widehat{C^g/S_g}$ has finitely many (>1) connected components.
- Recall: C^g/S_g is birational to the Jacobian $J = \mathrm{Jac}(C)$ of C.
- Using the lemma twice, we see that $\widehat{J}$ has finitely many (>1) connected components. (Both C^g/S_g and J are smooth.)
- But, as before, $(J(\mathbb{U}), +)$ is a divisible group acting transitively on the set of connected components of $\widehat{J}$. Contradiction ! □

The main theorem of Hrushovski-Loeser (a first version)

Theorem

Suppose $A = K \cup G$, where $K \subseteq \mathbf{VF}$ and $G \subseteq \Gamma_\infty$. Let V be a quasiprojective variety and $X \subseteq V \times \Gamma_\infty^n$ an A-definable subset.

Then there is an A-definable strong deformation retraction $H : \mathrm{I} \times \widehat{X} \to \widehat{X}$ onto a (Γ-internal) subset $\Sigma \subseteq \widehat{X}$ such that Σ A-embeds homeomorphically into Γ_∞^w for some finite A-definable w.

Corollary

Let X be as above. Then $\widehat{X}$ has finitely many definable connected components. These are all semi-algebraic and path-connected.

Proof.

Let H and Σ be as in the theorem. By *o*-minimality, Σ has finitely many def. connected components $\Sigma_1, \ldots, \Sigma_m$. The properties of H imply that $H_e^{-1}(\Sigma_i) = \widehat{X_i}$, where $X_i = H_e^{-1}(\Sigma_i) \cap X$ □

The main theorem of Hrushovski-Loeser (general version)

Theorem

Let $A = K \cup G$, where $K \subseteq \mathbf{VF}$ and $G \subseteq \Gamma_\infty$. Assume given:

1. *a quasiprojective variety V defined over K;*
2. *an A-definable subset of $X \subseteq V \times \Gamma_\infty^m$;*
3. *a finite algebraic group action on V (defined over K);*
4. *finitely many A-definable functions $\xi_i : V \to \Gamma_\infty$.*

Then there is an A-definable strong deformation retration $H : \mathrm{I} \times \widehat{X} \to \widehat{X}$ onto a (Γ-internal) subset $\Sigma \subseteq \widehat{X}$ such that

- *Σ A-embeds homeomorphically into Γ_∞^w for some finite A-definable w;*
- *H is equivariant w.r.t. to the algebraic group action from (3);*
- *H respects the ξ_i from (4), i.e. $\xi(H(t, v)) = \xi(v)$ for all t, v.*

Some words about the proof of the main theorem

- The proof is by induction on $d = \dim(V)$, **fibering into curves**.
- The fact that one may respect extra data (the functions to Γ_∞ and the finite algebraic group action) is essential in the proof, since these extra data are needed in the inductive approach.
- In going from d to $d+1$, the homotopy is obtained by a concatenation of four different homotopies.
- Only standard tools from algebraic geometry are used, apart from Riemann-Roch (used the proof of iso-definability of $\widehat{C}$).
- Technically, the most involved arguments are needed to guarantee the continuity of certain homotopies. There are nice specialisation criteria (both for the v- and for the g-topology) which may be formulated in terms of 'doubly valued fields'.

Berkovich spaces slightly generalised

A type $p = \mathrm{tp}(\overline{a}/A) \in S(A)$ is said to be **almost orthogonal to Γ** if $\Gamma(A) = \Gamma(A\overline{a})$.

- Let F a valued field s.t. $\Gamma_F \leq \mathbb{R}$.
- Set $\mathbb{F} = (F, \mathbb{R})$, where $\mathbb{R} \subseteq \Gamma$.
- Let V be a variety defined over F, and $X \subseteq V \times \Gamma_\infty^m$ an $\mathbb{F}$-definable subset.
- Let $B_X(\mathbb{F}) = \{p \in S_X(\mathbb{F}) \mid \text{p is almost orthogonal to } \Gamma\}$.
- In a similar way to the Berkovich and the HL setting, one defines a topology on $B_X(\mathbb{F})$.

Fact

If F is complete, then $B_V(\mathbb{F})$ and V^{an} are canonically homeomorphic. More generally, $B_{V\times\Gamma_\infty^m}(\mathbb{F}) = V^{an} \times \mathbb{R}_\infty^m$.

Passing from $\widehat{X}$ to $B_X(\mathbb{F})$

Given $\mathbb{F} = (F, \mathbb{R})$ as before, let $F^{max} \models \mathrm{ACVF}$ be maximally complete such that

- $\mathbb{F} \subseteq (F^{max}, \mathbb{R})$;
- $\Gamma_{F^{max}} = \mathbb{R}$, and
- $\mathbf{k}_{F^{max}} = \mathbf{k}_F^{alg}$.

Remark

By a result of Kaplansky, F^{max} is uniquely determined up to $\mathbb{F}$-automorphism by the above properties.

Lemma

The restriction of types map $\pi : \widehat{X}(F^{max}) \to S_X(\mathbb{F})$, $p \mapsto p|\mathbb{F}$ induces a surjection $\pi : \widehat{X}(F^{max}) \twoheadrightarrow B_X(\mathbb{F})$.

Remark

There exists an alternative way of passing from $\widehat{X}$ to $B_X(\mathbb{F})$, using imaginaries (from the lattice sorts).

The topological link to actual Berkovich spaces

Proposition

1. *The map* $\pi : \widehat{X}(F^{max}) \twoheadrightarrow B_X(\mathbb{F})$ *is continuous and closed. In particular, if* $F = F^{max}$*, it is a homeomorphism.*
2. *Let* X *and* Y *be* $\mathbb{F}$*-definable subsets of some* $V \times \Gamma_\infty^m$*, and let* $g : \widehat{X} \to \widehat{Y}$ *be continuous and* $\mathbb{F}$*-prodefinable.*

   Then there is a (unique) continuous map $\tilde{g} : B_X(\mathbb{F}) \to B_Y(\mathbb{F})$ *such that* $\pi \circ g = \tilde{g} \circ \pi$ *on* $\widehat{X}(F^{max})$*.*
3. *If* $H : \mathrm{I} \times \widehat{X} \to \widehat{X}$ *is a strong deformation retraction, so is* $\tilde{H} : \mathrm{I}(\mathbb{R}_\infty) \times B_X(\mathbb{F}) \to B_X(\mathbb{F})$*.*
4. $B_X(\mathbb{F})$ *is compact iff* $\widehat{X}$ *is definably compact.*

Remark

The proposition applies in particular to V^{an}*.*

The main theorem phrased for Berkovich spaces

Theorem

Let V be a quasiprojective variety defined over F, and let $X \subseteq V \times \Gamma_\infty^m$ be an $\mathbb{F}$-definable subset.

Then there is a strong deformation retraction

$$H : \mathrm{I}(\mathbb{R}_\infty) \times B_X(\mathbb{F}) \to B_X(\mathbb{F})$$

onto a subspace **Z** *which is homeomorphic to a finite simplicial complex.*

Topological tameness for Berkovich spaces I

Theorem (Local contractibility)

Let V be quasi-projective and $X \subseteq V \times \Gamma_\infty^m$ $\mathbb{F}$-definable. Then $B_X(\mathbb{F})$ is locally contractible, i.e. every point has a basis of contractible open neighbourhoods.

Proof.

- There is a basis of open sets given by 'semi-algebraic' sets, i.e., sets of the form $B_{X'}(\mathbb{F})$ for $X' \subseteq X$ $\mathbb{F}$-definable.
- So it is enough to show that any $a \in B_X(\mathbb{F})$ is contained in a contractible subset.
- Let H and $\mathbf{Z}$ be as in the theorem, and let $H_e(a) = a' \in \mathbf{Z}$. As $\mathbf{Z}$ is a finite simplicial complex, it is locally contractible, so there is $a' \subseteq W$ with $W \subseteq \mathbf{Z}$ open and contractible.
- The properties of H imply that $H_e^{-1}(W)$ is contractible.

Topological tameness for Berkovich spaces II

Here is a list of further tameness results:

Theorem

1. *If V quasiprojective and $X \subseteq V \times \Gamma_\infty^m$ vary in a definable family, then there are only finitely many homotopy types for the corresponding Berkovich spaces. (We omit a more precise formulation.)*
2. *If $B_X(\mathbb{F})$ is compact, then it is homeomorphic to $\varprojlim_{i \in I} \mathbf{Z_i}$, where the $\mathbf{Z_i}$ form a projective system of subspaces of $B_X(\mathbb{F})$ which are homeomorphic to finite simplicial complexes.*
3. *Let $d = \dim(V)$, and assume that F contains a countable dense subset for the valuation topology. Then $B_V(\mathbb{F})$ embeds homeomorphically into $\mathbb{R}^{2d+1}$ (Hrushovski-Loeser-Poonen).*

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